The Bundle Picture in Mechanics

by

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March 21, 1986

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Abstract.

We extend the work of Sternberg and Weinstein concerning the phase space of a particle in a Yang-Mills field. The phase spaces we investigate are Poisson manifolds which are also vector bundles over symplectic manifolds. Their Poisson structures are obtained by using a connection (Yang-Mills field) to splice together the canonical Poisson structure on the base symplectic manifold with a Lie-Poisson structure (Poisson structure on the dual of a Lie algebra) on the fiber. An intrinsic formula is given for the resulting Poisson bracket. Such Poisson structures are shown to arise on the normal bundle to a co-adjoint orbit. Other applications are given to the motion of an incompressible fluid with a free boundary, to a particle in a Yang-Mills field, and to a Yang-Mills plasma.

Dedication

This thesis is dedicated to Mary Montgomery, my mother. She went to a lot of trouble to bear me, and loved me as I was and not for what she wanted me to be or what I was going to be.

Acknowledgements

First, I would like to thank Judy my main squeeze and wife for her love and assistance. I am thankful she is still with me after the number of times I've come to bed at 2 in the morning thinking I've proved some great theorem, only to leave bed at 3, after tossing around for an hour while convincing myself that I was wrong.

I also thank my father, Roger Montgomery for his love and assistance, and for encouraging me to be a math nerd in the first place, for eventually accepting it when I decided to be a kayak bum instead of a math nerd, and for his hidden glee when I returned to my studies.

To David Barton, fellow graduate student who listened and gave me excellent advice at a crucial time.

To my adviser, Jerrold Marsden, I owe many thanks. He has nurtured my growth and tutored me in all aspects of research: not only in ideas and their workings, but also in more mundane areas such as writing, collaborating, and academic politics . I hope one day to approach being as good an adviser as he.

To Alan Weinstein. Alan has lent me his thoughtful ear many times, and I have benefitted greatly from numerous discussions with him.

To Tudor Ratiu: his unbridled enthusiasm got me over several humps. To Ted Courant who has been a sounding board and partner for many of the ideas here. To Steve Omohundro: our weekly discussions were a source of inspiration and delight.

There are a number of others who were crucial to my mathematical development as well as to this thesis, and I would like to thank them: Tepper Gill, Geoffery Mess, John Lott, Clifford Taubes, Harry Morrison, Darryl Holm, Jedrezj Sniatycki, Vincent Moncrief, Alberto Ibort, and Deborah Lewis.

I would also like to thank Chuck Stanley, Michael Schlax and Lars Holbek for making sure that I stayed a member of the Four Stooges in their Skinny Boats.

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Introduction

This thesis presents contributions to "the bundle picture" for Poisson manifolds and then applies them to obtain Hamiltonian formulations for three systems of physical interest. The painting of the bundle picture was begun by Sternberg [1977] and Weinstein [1978] in their investigations of the symplectic geometry of the phase space of a classical colored particle in an external Yang-Mills field. The phase space of such a particle is a vector bundle over standard single particle phase space with fiber the vector space of color charges. The Poisson bracket on the phase space depends on the Yang-Mills potential (connection). The particle's dynamics is governed by Hamilton's equations with the Hamiltonian being the standard kinetic energy of the particle. The resulting equations of motion are known as Wong's equations.

Abstractly, the starting point for the bundle picture is T^*B, the cotangent bundle of a principal G-bundle $B \to X$. In the colored particle example, X represents the underlying space (or space-time) through which the particle travels. The structure group G acts on T^*B by canonical transformations and has momentum map $T^*B \to \mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G. In the example of the colored particle, $\mathfrak{g}^*$ represents the space of color charges. The quotient T^*B/G is a Poisson manifold (called the Poisson reduced space). Its symplectic leaves are the spaces investigated by Weinstein [1978]. T^*B/G is a vector bundle over X with fiber $T^*_xX \times \mathfrak{g}^*$. Making T^*B/G into a vector bundle over T^*X requires the choice of a connection A

for the principal bundle B.

Instead of using the connection A to make T^*B/G into a vector bundle over T^*X, we can use it to split TB, and so by duality T^*B, into horizontal + vertical. This splitting can be thought of as an isomorphism $\Phi_A: B^\# \times \mathfrak{g}^* \to T^*B$, where $B^\#$ is the pull-back bundle of B to T^*X ($B^\#$ is a principal G-bundle over T^*X). Φ_A intertwines the G action on T^*B with the "diagonal" G action ((action as the structure group of $B^\#$) x (co-adjoint action)) on $B^\# \times \mathfrak{g}^*$. The quotient of $B^\# \times \mathfrak{g}^*$ by this diagonal action is written $B^\# \times_G \mathfrak{g}^*$ or $Ad^*(B^\#)$, and is called the co-adjoint bundle over T^*X associated to $B^\#$. (This is the standard associated bundle construction.) It is a vector bundle over T^*X with fiber $\mathfrak{g}^*$. If we take X to be space or space-time and G to be SU(3) then this co-adjoint bundle is the phase space of our colored particle. The quotient map $[\Phi_A]: Ad^*(B^\#) \to T^*B/G$ is an isomorphism. We call this isomorphism the "minimal coupling procedure" since its coordinate expression is precisely the physicists' minimal coupling procedure. We put a Poisson structure on $Ad^*(B^\#)$ by using the minimal coupling procedure to pull back the Poisson structure on T^*B/G. The symplectic leaves are then the symplectic manifolds introduced by Sternberg [1977].

The first result of this thesis is an intrinsic formula for the Poisson bracket on the co-adjoint bundle which is presented at the end of §1.1. Schematically, this formula reads

{F,G} = canonical bracket on T*X

+ curvature (fiber-base interaction) term

+ Lie-Poisson bracket for the fiber $\mathfrak{g}^*$ [PB.1].

This result is not new, being first proved by local calculation in Montgomery, Marsden, and Ratiu [1984]. However, the proof provided in this thesis (corollary to Theorem 1 of §1.2) is new, being coordinate-free, and it also leads to an interesting generalization of the formula [formula [PB1] of §1.2].

The coordinate free proof relies on "Poissonizing" a symplectic construction of Sternberg's [1977]. The result of this "Poissinization" is a connection-dependent Poisson bracket on (a neighborhood of the zero section of) any co-adjoint bundle over an **arbitrary** symplectic manifold. In other words, we can drop the restriction that the base space of the co-adjoint bundle be a cotangent bundle. The payments for this increase of generality are that

(i) the Poisson structure may blow-up away from the zero section
(ii) an extra term may have to be added to formula [PB.1].

(These two payments are related in that (i) implies (ii).) The investigation of this generalization of Sternberg's construction is carried out in §1.2, the main result being Theorem 1 of that section.

We present four applications of the bundle picture. In the first application (§1.3) we investigate Poisson structures on the normal bundle of a co-adjoint orbit in the dual of a Lie algebra. We only consider reductive co-adjoint orbits (all co-adjoint orbits for compact groups are reductive). Normal bundles of such orbits inherit two Poisson structures, one induced by a naturally arising connection, the other induced by an exponential map of the co-adjoint orbit into the dual Lie algebra. An application of Theorem 1 of §1.2 shows that these two Poisson structures are actually the same. As an added bonus, Theorem 1 allows us to understand the singularities of this Poisson structure.

The other three applications are motivated by physics and are presented in the second part of the thesis. The phase space for each of these three examples can be obtained as the reduction of the phase spaces T*B where the configuration space B is a principal bundle. Thus in each of these examples reduced Poisson brackets (depending on a connection) of the form [PB.1] are derived.

The first physical application (§2.1) is to the dynamics of the colored particle mentioned above. There are four formulations of the dynamics of such a particle: the symplectic formulation of Sternberg [1977], the symplectic formulation of Weinstein [1978], the geodesic formulation of Kaluza-Klein and Kerner [1968], and the formulation of the physicist Wong [1970]. Weinstein showed that his formulation was equivalent to Sternberg's. Sniatycki [1979] showed that these two formulations are equivalent to that of Kaluza-Klein and Kerner. Montgomery [1984] showed how these formulations are equivalent to that of Wong. §2.1 is a rewriting of Montgomery [1984], the main change being that care is taken to point out the differences between the relativistic and non-relativistic formulations.

The second physical applications (S2.2) is to a plasma of colored particles in the self-consistent field approximation. We use the approach of Marsden-Weinstein [1982] in their investigation of the Hamiltonian formulation of the Maxwell-Vlasov equations, i.e. of an Abelian plasma. The action of the gauge group on the unreduced plasma phase space of {Yang-Mills potentials} x {electric fields} x {plasma densities} is more complicated in the non-Abelian case. This adds two complications to the Hamiltonian formulation of a non-Abelian plasma. The first is that the momentum map is more complicated. However, it is calculable by standard methods. This is done in S1.4. The second complication is that the quotient space of connections modulo the gauge group is much more complicated topologically in the non-Abelian case, and so does not admit a global coordinatization. Thus we rely on a local choice of gauge in order to explicitly write down the Poisson brackets on the reduced space.

The third and final physical application (S2.3) is to an understanding of the Hamiltonian structure for the flow of an incompressible fluid with free boundary and surface tension, i.e. a water drop. There are two versions of the Poisson bracket, one corresponding to the reduced cotangent bundle T^*B/G, and one to the co-adjoint bundle $Ad^*(B^\#)$. Both generalize the canonical brackets which Zakharov [1968] found for the irrotational case. These brackets were found useful by Lewis et al. [1985] in their work on stability of rotating water drops.

Other applications and future directions

Alan Weinstein [1985] generalized his [1978] construction in order to investigate a symplectic model for the principal series of group representations. The Poisson manifold T^*B/G mentioned above is isomorphic to the Poisson reduction at 0 of the Poisson manifold $T^*B \times \mathfrak{g}^*$; in symbols: $T^*B/G \simeq (T^*B \times \mathfrak{g}^*)_0$. Weinstein's generalization was to replace $\mathfrak{g}^*$ with an arbitrary Poisson manifold P which admits a momentum map $J_P : P \to \mathfrak{g}^*$ (in the case above where $P = \mathfrak{g}^*$, J_P is minus the identity map). Call the pair (P, J_P) a **Poisson G-manifold**. One of the main results in Weinstein [1985] is that the assignment $(B,G,P,J_P) \mapsto (T^*B \times P)_0$ is a functor from the category of principal G-bundles and Poisson G-manifolds to the category of Poisson manifolds.

Carinena and Ibort [1985] have applied the bundle picture in order to give a canonical interpretation of ghost fields and of the B.R.S. transformations which are important tools in the quantization of Yang-Mills fields. Let B be the space of (irreducible) connections on a principal bundle, G be the gauge group for the principal bundle and X = B/G. Take A to be the Couloumb connection on $B \to X$. Then $\square_A$: $T^*B \simeq B^\# \times \mathfrak{g}^*$. Carinena and Ibort have shown that the correct interpretation of the $\mathfrak{g}^*$ factor is as the set of ghost fields. The B.R.S. transformations are the infinitesimal transformations on $B^\# \times \mathfrak{g}^*$

corresponding to the diagonal action of G.

Guillemin and Uribe [1985] used the bundle picture in an investigation of spectral problems on S^4 and on S^2. They studied the spectral asymptotics of the operator covariant Laplacian plus potential. This operates on spaces of sections of vector bundles over S^4 or S^2. These spectral problems can be thought of as quantum versions of the classical Wong's equations for these bundles. Guillemin and Uribe found Montgomery [1984] helpful in understanding these spectral problems.

The covariant Laplacian is the operator $D_A{}^*D_A$ where D_A is the covariant derivative associated to the connection A. For the problem over S^4, Guillemin and Uribe took A to be the standard connection for the quaternionic Hopf fibration. Guillemin, Uribe and the author are currently working on extending their [1985] results to the case where A is any self-dual connection for this fibration. This involves solving Wong's equations with the Yang-Mills field being an arbitrary self-dual connection on S^4 with instanton number one. So far, we have been able to show that this problem is completely integrable.

Atiyah and Hitchin [1985] have investigated the dynamics of monopoles in the adiabatic limit. It would be interesting to couple Atiyah's equations with Wong's equations for a particle in a monopole field. It is conceivable that the resulting approximate model for a Yang-Mills plasma would be exactly solvable.

§1.1 Reducing the Cotangent Bundle of a Principal Bundle.

Introduction to §1.1

This section is a review and summary of existing results concerning the two isomorphic Poisson reductions of the cotangent bundle of a principal bundle by its structure group. One of these reductions depends on a connection and the other does not. We describe both of the resulting Poisson structures in coordinates and in coordinate-free language.

The first papers in this area were in the symplectic category. The seminal papers are Sternberg [1977] where the connection-dependent symplectic reduced space was described, and Weinstein [1978] where the "intrinsic" symplectic reduced space was described. The present section follows the lines of Montgomery [1984] which "Poissonized" these seminal papers and connected them with the physics literature. This section ends with a coordinate-free formula, first stated in Montgomery, Marsden, and Ratiu [1985], for the connection-induced brackets. This bracket has the schematic form:

{F,G} = canonical bracket on base
+ curvature (interaction) term
+ Lie-Poisson bracket on fibres.

Let $\pi: B \rightarrow X$ be a principal right G-bundle. Thus G acts on B on the right. The cotangent lift of the G action to T^*B is a canonical action. The "intrinsic" Poisson reduced space is simply the quotient T^*B/G. Think of functions on the quotient as G-invariant functions on T^*B. Their Poisson bracket is just their usual Poisson bracket on T^*B, which is another G-invariant function, so can be thought of as a function on the quotient.

To describe the other Poisson structure requires more machinery. The G-action on T^*B has equivariant momentum map $\sigma^*: T^*B \to \mathfrak{g}^*, \sigma^*(\alpha_b) = \sigma_b^*(\alpha_b)$, where $\sigma_b: \mathfrak{g} \to T_bB$ is the infinitesimal generator of the G-action on B:

$$\sigma_b(\xi) = d/d\lambda|_{\lambda=0}(b\exp\lambda\xi),\ \xi \in \mathfrak{g}.$$

One easily sees that $\sigma^{*-1}(0) = V^\circ$, the annihilator of the vertical subbundle $V = \ker T\pi \subset TB$. V° is a realization of the pull-back bundle $B^\#$ of B by the cotangent projection $\tau_X: T^*X \to X$. This means that we have a principal G-bundle map:

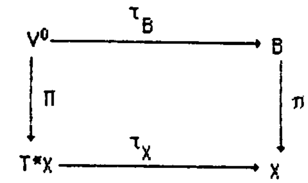


diagram 1.1.1

The map $\tau_B: V^0 \to B$ denotes the restriction of the cotangent projection $T^*B \to B$ to the G-invariant subbundle V^0. The projection $\Pi: V^0 \to T^*X$ is defined by

$$\langle \Pi(\alpha_b), T\pi_b \cdot v_b \rangle = \langle \alpha_b, v_b \rangle,\ \text{for } \alpha_b \in T^*_bB, v_b \in T_bB.$$

[To see that $\Pi(\alpha_b)$ is a well defined element of T^*_xX, $x = \pi(b)$, note that α_b annihilates $\ker T\pi_b$ so that $\langle \alpha_b, v_b \rangle = \langle \alpha_b, \tilde{v}_b \rangle$ provided that $T\pi_b \cdot v_b = T\pi_b \cdot \tilde{v}_b$, and note that π is a submersion, so that any vector in T_xX can be written in the form $T\pi_b \cdot v_b$.]

Let A be a connection on B. So

$$A_b: T_bB \to \mathfrak{g},\ b \in B$$

is a $\mathfrak{g}$-valued one form on B satisfying

$A_b \cdot \sigma_b$ = identity on $\mathfrak{g}$, and $A_{bg} = Ad_{g^{-1}} \cdot A_b \cdot TR_{g^{-1}}$.

A splits TB into the vertical and horizontal subbundles over B:

$T_bB = V_b \oplus H_b$, $V_b = \ker T\pi_b$, $H_b = \ker A_b$, for $b \in B$.

The dual splitting can be thought of as an isomorphism:

$\Phi = \Phi(A)$: $B^\# \times \mathfrak{g}^* \longrightarrow V^\circ \oplus H^\circ = T^*B$;

where $V^\circ = B^\#$ = Annihilator of V

and H° = Annihilator of H.

Φ is given by

$\Phi(A)(\alpha_b, \mu) = \alpha_b + A_b{}^*\mu$.

The inverse of Φ is given by

$\Phi(A)^{-1}(\beta_b) = (\beta_b - \sigma_b{}^* A_b{}^* \beta_b, \sigma_b{}^* \beta_b)$.

The equivariance property of A implies that Φ is a G-equivariant isomorphism, where the action on T*B is the cotangent lift action discussed above and where the action on $B^\# \times \mathfrak{g}^*$ is the diagonal one:

$(\alpha_b, \mu) \cdot g = (TR_{g^{-1}}{}^* \alpha_b, Ad_g{}^* \mu)$.

Note that the action on the $B^\#$ factor is just the restriction of the G action on T*B to $B^\# \subset T^*B$. Also note that we have the commutative diagram

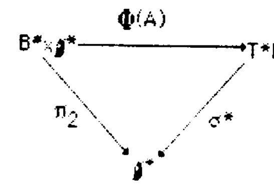


diagram 1.1.2

Using Φ, we can pull back the canonical symplectic form $\omega_B = -d\theta_B$ on T*B to obtain a connection-dependent symplectic form on $B^\# \times \mathfrak{g}^*$. A straightforward calculation yields:

$\Phi^*\theta_B = \Pi^*\theta_X + \langle \mu, A^\# \rangle$

where θ_X is the canonical one-form on T*X, where $A^\# = \tau_B{}^* A$ is the pull back connection on $B^\#$, and where $\langle \mu, A^\# \rangle$ denotes the one-form $(\alpha_b, \mu) \longmapsto \langle \mu, A^\#(\alpha_b) \rangle$. It follows that

$$\Phi^*\omega_B = \Pi^*\omega_X - d\langle \mu, A^\# \rangle . \qquad [1.1.1]$$

where $\omega_X = -d\theta_X$ is the canonical two-form on T^*X.

Remark. Basically the same symplectic form was used by Sternberg as an intermediate step in [1977]. In his set-up $\mathfrak{g}^*$ was replaced by a symplectic G-manifold with momentum map $J: F \to \mathfrak{g}^*$. The form he used was $\Pi^*\omega_X + d\langle J, A^\# \rangle$. Since the momentum map for the *left* co-adjoint action on $\mathfrak{g}_+^*$ is $\mu \mapsto -\mu$, we are in essence using the same form as he.

We now form the Poisson reduced space as before by dividing through by G. This yields the following commutative diagram of Poisson maps:

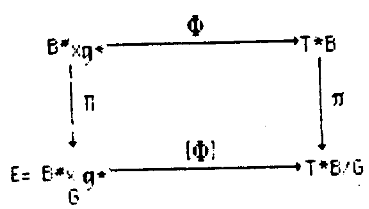


diagram 1.1.3

The horizontal arrows are Poisson isomorphisms. The quotient of $B^\# \times \mathfrak{g}^*$ by G is an associated vector bundle to $B^\#$. It is called the **co-adjoint bundle**, written $B^\# \times_G \mathfrak{g}^*$ or $Ad^*(B^\#)$, and is a vector bundle over T^*X with fibre $\mathfrak{g}^*$. As proved in Montgomery [1984] and in Theorem 1 of the next section, the symplectic leaves of $Ad^*(B^\#)$ are the spaces $B^\# \times_G \Theta$, $\Theta \subset \mathfrak{g}^*$ is a co-adjoint orbit, which were constructed by Sternberg [1977]. Note that the construction of $Ad^*(B^\#)$ and its projection onto T^*X are **independent** of the choice of the connnection A. However, its Poisson structure depends strongly on the choice of A, through the splitting Φ.

The Poisson structure of the other quotient, T^*B/G, is clearly connection-independent. However, without a connection there is no projection from T^*B/G to T^*X. The connection A defines such a projection through its horizontal lift. Let

$$h_b: T_xX \to T_bB$$

denote the horizontal lift defined by A: $\operatorname{im} h_b = H_b$, and $T\pi_b \circ h_b$ =identity. Then

$$[\alpha_b] \mapsto h_b^*\alpha_b$$

is the projection. Here $[\alpha_b] \in T^*B/G$ denotes the equivalence class, i.e. the G orbit, containing $\alpha_b \in T^*B$. Since $h_{bg} = TR_g \circ h_b$, the covector $h_b^*\alpha_b \in T_x^*X$ is independent of which representative, α_b of this equivalence class is picked. Summarizing

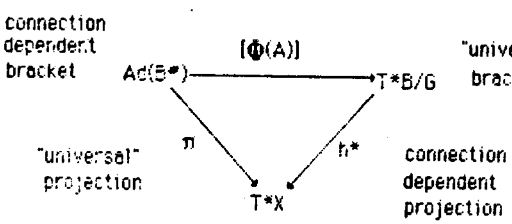


diagram 1.1.4

The map $[\Phi(A)]$ will be called the **minimal coupling procedure**, for reasons which will become clear in the following local discussion.

Local Structure.

It will be necessary to understand these Poisson structures locally as well as globally. A local trivialization $B_U \simeq U \times G$ along with coordinates x^μ, $\mu = 1,\ldots,n$ on $U \subset X$ induces coordinates $(x^\mu, p^{can.}{}_\alpha, Q_a)$ or simply $(x, p^{can.}, Q) \in \mathbb{R}^n \times \mathbb{R}^{n*} \times \mathfrak{g}^*$ on T^*B/G, as follows. Since $B_U \simeq U \times G$ we have $T^*(B_U) \simeq T^*(U \times G) \simeq T^*U \times T^*G$, as symplectic

manifolds. Then as Poisson manifolds $T^*B/G \supset T^*B_U/G \simeq (T^*U \times T^*G)/G = T^*U \times (T^*G/G) \simeq T^*U \times \mathfrak{g}^*$. The G action on T^*G is the **right** action. This corresponds to the fact that the $\mathfrak{g}^*$ factor will have its + Lie-Poisson structure. The x, $p^{can.}$ are the usual canonical coordinates on T^*U. The Q_a are linear coordinates on $\mathfrak{g}^*$, so depend on a choice of basis for $\mathfrak{g}$. Let $c^d{}_{ab}$ be the structure constants of $\mathfrak{g}$ relative to this choice of basis. The brackets on T^*B_U/G are then given by

$$\{x^\mu, p^{can.}{}_\beta\} = \delta^\mu{}_\beta$$

$$\{Q_a, Q_b\} = Q_d c^d{}_{ab}$$

all other brackets zero.

In the case where B is the principal bundle for a Yang-Mills theory over space X (or space-time), T^*B/G is interpreted as the phase space for a classical quark under the influence of an external Yang-Mills field. The x^μ are the quark's spacetime coordinates. The $p^{can.}{}_\mu$ are its canonical (as opposed to physical, or kinetic) momenta. The Q_a are its color charges. This example is presented in §2.1.

The coordinatization of $Ad^*(B^*)$ goes as follows. The local trivialization $B_U \simeq U\times G$ induces the local trivialization $B^*_{T^*U} \simeq T^*U \times G$ of the pull- back bundle and induces the local vector bundle trivializations of $Ad^*(B)_U \simeq U\times\mathfrak{g}^*$ and $Ad^*(B^*)_{T^*U} \simeq T^*U\times\mathfrak{g}^*$. Again, the coordinates x^μ induce coordinates on $T^*U \subset T^*X$, but this time these cotangent coordinates will be written (x,p) instead of $(x, p^{can.})$. The coordinitization of $\mathfrak{g}^*$ is the same. Then $(x,p,Q) \in \mathbb{R}^n\times\mathbb{R}^{n*}\times\mathfrak{g}^*$ are coordinates on $Ad^*(B^*)$.

With respect to these trivializations, the coordinate description of diagram 1.1.3 is

$$\begin{array}{ccc} (x,p,g,Q^*) & \xmapsto{\;\Phi(A)\;} & (x, p^{can.}, g, Q) \in \mathbb{R}^n\times\mathbb{R}^{n*}\times G\times\mathfrak{g}^* \\ \downarrow & & \downarrow \\ (x,p,Q) & \xmapsto[\;[\Phi(A)]\;]{} & (x, p^{can.}, Q) \in \mathbb{R}^n\times\mathbb{R}^{n*}\times\mathfrak{g}^* \end{array}$$

[diagram 1.1.3a]

where

$$Q^* = Ad_g^{\,*}Q\,,$$

and

and

$$p^{can.} = p + Q\cdot A\,,$$

and " · " denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. (In the top right hand corner of the diagram we have **right** trivialized $T^*G \simeq G\times\mathfrak{g}^*$.) This last equation is the classical relationship between the canonical momenta, $p^{can.}$ and the physical, or kinetic momenta p which is at the foundation of the minimal coupling procedure for gauge theories, including electromagnetism. This is why we call $[\Phi(A)]$ the minimal coupling procedure.

Since $[\Phi(A)]$ is a Poisson map, the local expression for it can be used to calculate the (x, p, Q) brackets from the $(x, p^{can.}, Q)$ brackets. One finds

$$\{x^\mu, p_\beta\} = \delta^\mu{}_\beta$$

$$\{p_\mu, p_\beta\} = Q_a F^a{}_{\mu\beta}$$

$$\{Q_a, Q_b\} = Q_d c^d{}_{ab}$$

$$\{p_\mu, Q_a\} = -Q_d c^d{}_{ab} A^b{}_\mu$$

all other brackets zero.

Here $A^b{}_\mu$ is the local expression for the connection A, and $F^a{}_{\mu\beta}$ is the local expression for its curvature.

Relationship with earlier versions of cotangent reduction.

There are a number of works on the symplectic reduction of the cotangent bundle T^*B of a principal G-bundle B. In primitive form, these works date back at least to Smale [1970]. Recall that the symplectic reduced space at $\mu \in \mathfrak{g}^*$, denoted $(T^*B)_\mu$, is the quotient space $\sigma^{*-1}(\mu)/G_\mu$ where G_μ is the isotropy group of μ. One of the goals of these earlier works was to determine when the symplectic reduced space was the cotangent bundle of the base space X.

<u>**Cotangent bundle reduction theorem**</u> [Satzer-Marsden-Kummer]. **Suppose that $G = G_\mu$. Then the symplectic reduced space $(T^*B)_\mu$ is symplectically isomorphic to T^*X with the symplectic form $\omega_X - \langle\mu, F^\#\rangle$. Here $F^\#$ denotes the curvature of the connection $A^\#$ on the bundle $B \to T^*X$.**

This theorem is due to Satzer [1977] for the case where G is Abelian, in which case G_μ always equals G. Abraham and Marsden [1978] generalized Satzer's result to the case $G_\mu = G$, not necessarily Abelian. For example, if $\mu = 0$ then G_μ always equals G. The realization that the term subtracted from the canonical two-form ω_X can be written $\langle\mu, F^\#\rangle$ is due to Kummer [1981].

We will now reprove this theorem from our point of view. From commutative diagram 1.1.2 we know that $\Phi(A)$ maps $\sigma^{*-1}(\mu)$ isomorphically onto $B^\# \times \{\mu\}$. This map is $\alpha_b \mapsto (\alpha_b - A_b^*\sigma^*\alpha_b, \sigma^*\alpha_b = \mu)$. Being the restriction of a G-equivariant map to a G_μ-equivariant submanifold, it is G_μ-equivariant. So, if $G = G_\mu$ we can divide through by G, obtaining $[\Phi(A)]: \sigma^{*-1}(\mu)/G_\mu \to B^\# \times_G \{\mu\} = B^\#/G \times \{\mu\} \simeq T^*X \times \{\mu\}$. Dropping the constant factor μ, we have the isomorphism: $[\alpha_b] \mapsto [\alpha_b - A_b^*\sigma^*\alpha_b]$. This is the isomorphism used by Abraham-Marsden [1978, p.300-301] and Kummer [1981, p.283, eq.3].

To calculate the symplectic form ω on T^*X induced by this isomorphism, recall the definition of the reduced symplectic form on $\sigma^{*-1}(\mu)/G_\mu$. It is defined by the condition that its pull-back to $\sigma^{*-1}(\mu) \subset T^*B$ equals the restriction of ω_B to $\sigma^{*-1}(\mu)$. Since $\Phi(A)^*\omega_B = \pi^*\omega_X - d\langle\mu, A\rangle$, the form ω on T^*X is defined by

$$\pi^*\omega = i^*(\pi^*\omega_X - d\langle\mu, A\rangle)$$

where i is the inclusion $B^\# \to B^\# \times \{\mu\} \hookrightarrow B^\# \times \mathfrak{g}^*$. Now $i^*d\langle\mu, A^\#\rangle = \langle\mu, dA^\#\rangle = \langle\mu, F^\#\rangle - \langle\mu, [A^\#, A^\#]\rangle$. Since $G_\mu = G$, it follows that $\langle\mu, [\xi, \gamma]\rangle = 0$ for all $\xi, \gamma \in \mathfrak{g}^*$, and hence that $\langle\mu, [A^\#, A^\#]\rangle = 0$. Thus $\omega = \omega_X - \langle\mu, F^\#\rangle$, agreeing with Kummer's result. (Kummer's form is actually $d\theta_X + \langle\mu, F^\#\rangle$. This sign difference is accounted for by the fact that the symplectic forms he uses on T^*X and T^*B are $+d\theta_X$ and $+d\theta_B$, which are the negatives of the symplectic forms which we use.)

Remarks.

1. The term $\langle\mu,F^*\rangle$ is called a "magnetic term" (see S2.1 of this thesis for the reason behind this terminology). Kummer [1981] notes that it has an obstruction-theoretic interpretation. It is a closed two-form, so represents a cohomology class on T^*X. This cohomology class is zero if and only if a connection A for $B \to X$ can be found such that $[\Phi(A)]$ induces the standard symplectic form on T^*X.

2. In the general circumstance where $G_\mu \neq G$, the isomorphism $(T^*B)_\mu \simeq T^*X$ generalizes to become a symplectic embedding of $(T^*B)_\mu$ into $T^*(B/G_\mu)$. Montgomery [1984] sketched how to obtain this embedding from the viewpoint of this thesis. The basic observation is that there is a natural isomorphism $B/G_\mu \simeq B\times_G\mathcal{O} \subset Ad^*(B)$ where $\mathcal{O}$ is the co-adjoint orbit through μ.

3. If the G action on B is not free, then $B \to B/G = X$ is no longer a principal bundle (and in general no longer even a manifold), but there is still a version of the cotangent bundle reduction theorem. This is due to Montgomery [1983] and applies, for example, to the case $B = \mathbb{R}^3$, $G = SO(3)$ with the standard action.

4. In case G is a torus and the manifold being reduced is compact (instead of T^*B), it is still true that the reduced symplectic form varies linearly with $\mu \in \mathfrak{g}^*$ for μ a regular value of the momentum map. This is the main result of Duistermaat and Heckman [1982]. Their result in turn is intimately related to the important and celebrated fact (Atiyah [1982], [1983], Guillemin and Sternberg [1982]) that the image of the momentum map is a convex polyhedron in $\mathfrak{g}^*$.

Intrinsic Bracket Formula.

We will end this section by stating the intrinsic bracket formula on $Ad^*(B^\#)$. We first introduce the necessary notation.

$$e = [\alpha,\mu] \in Ad^*(B^\#), \qquad \alpha \in B^\#_b \subset T^*_bB, \quad \mu \in \mathfrak{g}^*$$

$$p = \Pi(e) = \Pi(\alpha) \in T^*_xX, \qquad x = \pi(b).$$

We have used Π to denote the two different projections, $Ad^*(B^\#) \to T^*X$ and $B^\# \to T^*X$ but this should not cause any confusion. The connection A induces the pull-back connection $A^\# = \tau_B{}^*A$ on $B^\#$ and on its associated vector bundle $Ad^*(B^\#)$. A section of $Ad^*(B^\#)$ which is covariantly constant through e will be denoted by h. Covariant constancy uniquely determines the section's differential at p. In fact

$$T_ph = h_e: T_p(T^*X) \to T_eAd^*(B^\#),$$

the horizontal lift with respect to the connection $A^\#$ on $Ad^*(B^\#)$.

The curvature of $A^{\#}$ will be denoted $F^{\#}$ ($= \tau_B{}^*F$). We think of $F^{\#}$ as an $Ad(B^{\#}) = B^{\#}\times_G\mathfrak{g}$-valued two-form on T^*M. Note that $Ad(B^{\#})$ is the dual vector bundle to $Ad^*(B^{\#})$.

If f is a function on $Ad^*(B^{\#})$, its vertical differential at e, written $d_V f(e)$, is the element of $Ad(B^{\#})_p$ defined by

$$(d/d\lambda)\big|_{\lambda=0} f(e+\lambda v) = \langle v, d_V f(e)\rangle \text{, for } v \in E_p .$$

Here $\langle\ ,\ \rangle$ denotes the natural pairing between $Ad^*(B^{\#})$ and $Ad(B^{\#})$. Finally, note that the fibres of $Ad(B^{\#})$ have a natural Lie algebra structure, also denoted $[\ ,\ ]$ (see the appendix for details).

We can now state the <u>intrinsic formula</u> for the Poisson brackets on $Ad^*(B^{\#})$.

$$\{f,g\}(e) = \{h^*f, h^*g\}_{T^*X}(p) + \langle e, F^{\#}(X_{h^*f}, X_{h^*g})\rangle + \langle e, [d_V f, d_V g]\rangle \qquad \text{[PB.1]}$$

Here, $\{\ ,\ \}_{T^*X}$ denote the canonical brackets on T^*X and X_{h^*f} denotes the Hamiltonian vector field on T^*X corresponding to h^*f: $dy(X_{h^*f}) = \{y, h^*f\}_{T^*X}$ for y a function on T^*X. This formula will be proved as a corollary to Theorem 1 of the next section. It was originally proved by local calculation in Montgomery, Marsden, and Ratiu, [1985].

§1.2. The Intrinsic Poisson Bracket Formula: Generalization and Proof.

In this section we investigate a generalization, due to Sternberg [1977] in the symplectic case, of the Poisson manifold discussed in the previous section. Our main result is formula [PB.2] of Theorem 1, which generalizes the intrinsic formula of the previous section.

Let S be a symplectic manifold with symplectic form ω_S. Sternberg [1977] investigated fiber bundles E over S with structure group G whose fibres F are symplectic G-manifolds. Let B be the associated principal bundle to E, so that $E = B\times_G F$, and let A be a connection on B. Using the connection, Sternberg showed how to put the symplectic structures of the base and fiber together in order to obtain a pre-symplectic structure on the total space E. We replace the symplectic fiber F with the Poisson manifold $\mathfrak{g}^*$, the dual Lie algebra to the structure group G. E then becomes the associated vector bundle known as the co-adjoint bundle to B: $E = B\times_G\mathfrak{g}^* = Ad^*(B)$ and Sternberg's construction gives a Poisson structure (with possible singularities) on E. The symplectic leaves of E are Sternberg's spaces, with fibre F a co-adjoint orbit in $\mathfrak{g}^*$.

The other theorems and corollaries of this section besides Theorem 1 are concerned with the deviation of the general bracket [PB.2] from the bracket of the previous section. This deviation is contained in a single term, which we call the "cross-curvature term".

Consider the two-form

$$\Omega = \pi^*\omega_S - d\langle\mu,A\rangle \qquad [1.2.1]$$

on $B\times\mathfrak{g}^*$. Here $\langle\mu,A\rangle$ denotes the one-form $(b,\mu)\mapsto\langle\mu,A(b)\rangle$. Ω is closed. It is also invariant under the diagaonal right G action $(b,\mu)\mapsto(bg,Ad_g{}^*\mu)$ since A transforms according to $A(bg) = Ad_{g^{-1}}{}^*A(b)^*TR_{g^{-1}}$. Note that formula [1.1.1] of the previous section states that $\Omega = \Phi^*\omega_B$ satisfies [1.2.1], where S is replaced by T^*X, B by B^*, and A by A^*. (This two-form was used by Sternberg in [1977]. Guillemin and Sternberg investigated this form in detail [1985, ch. 40, prop.40.1] and proved the first part of Theorem 1. In remark 5 below we show how to obtain Ω as the pull-back of a form on T^*B.)

In Theorem 1 below we show that Ω is nondegenerate in a neighborhood U of $B\times\{0\}$. To describe this neighborhood U, as well as to state the theorem, we will develop some more notation. We begin by fixing the names of elements in the various spaces, and their projections:

$(b,\mu) \in B\times\mathfrak{g}^*$,

$e = p(b,\mu) = [b,\mu] \in B\times_G\mathfrak{g}^* = E$

$x = \pi(b) = \pi(e) = \pi(p(b,\mu)) \in S$

$p: B\times\mathfrak{g}^* \to E$,

$\pi: B,\ E \to S$.

Note that we use the same letter π to denote the two bundle projections onto S, but this should not cause any confusion.

Let F denote the curvature of A and h the horizontal lift for A. F will be thought of as either a two-form on S with values in $E^* = B\times_G\mathfrak{g}$ or an equivariant two-form on B with values in $\mathfrak{g}$, whichever is more convenient. The relation between the two is:

$$\langle e,F(x)(v,w)\rangle = \langle\mu,F(b)(hv,hw)\rangle .$$

On T_xS consider the antisymmetric bilinear form

$$\omega_e = \omega(x) - \langle e,F\rangle \qquad [1.2.2].$$

(ω_e is an element of $\Lambda^2T_x{}^*S$, and not a section of Λ^2T^*S, that is, it is not a two-form. However, we will sometimes abuse language and call ω_e a two-form. If one wished to make it into a legitimate two-form defined on a neighborhood of x, one would have to replace the element e of E with a local section $y\mapsto e(y)$ of $Ad^*(B) = E$.). **The neighborhood U on which Ω is nondegenerate consists of those (b,μ) for which ω_e is nondegenerate.** Fix x. Then the set of e in E_x such that ω_e is non-degenerate (that is the set $p(U)\cap E_x$) is a Zariski-open subset of E_x, since it is complement of the solution set of the polynomial equation $\det\omega_e = 0$. See the picture at the end of the next section for a class of examples where the solution set of the equation $\det\omega_e = 0$ is the union of a finite number of affine planes in E_x. When $\dim(S) = 4$ and $\dim(E_x) = 2$ examples can be given such that this solution set is any given conic section not containing the origin.

Let the inverse of ω_e (as a map $T_xS \to T^*_xS$) be denoted by $J_e: T^*_xS \to T_xS$. For $e \in p(U)$ and $\alpha \in T^*_xS$ define $Z_e(\alpha)$ by

$$J_e(\alpha) = J_0(\alpha) + Z_e(\alpha).$$

Note that $J_0(\alpha)$ is the inverse of $\omega = \omega_0$. Now

$$\begin{aligned}\omega_e(Z_e(\alpha), \cdot\,) &= \omega_e(J_e(\alpha), \cdot\,) - \omega_e(J_0(\alpha), \cdot\,)\\ &= \alpha - \omega_0(J_0(\alpha), \cdot\,) + \langle e, F(J_0(\alpha), \cdot\,)\rangle\\ &= \langle e, F(J_0(\alpha), \cdot\,)\rangle\end{aligned}$$

or

$$Z_e(\alpha) = J_e(\langle e, F(J_0(\alpha), \cdot\,)\rangle)$$

Finally, set

$$Z_f = Z_e(df)\ .$$

so that

$$\omega_e(Z_f, \cdot\,) = \langle e, F(X_f, \cdot\,)\rangle$$

where $X_f = J_0(df)$ denotes the Hamiltonian vector field for f with respect to ω, (<u>not</u> ω_e).

The rest of the notation needed to state Theorem 1 is the same as was needed to state the global bracket formula at the end of the previous section: $h: S \to E$ denotes a local section through $e = h(x)$ which is covariantly constant at s ($Dh(x) = 0$), the curvature F of A is to be interperted as a two-form on S with values in $E^* \simeq Ad(B)$, and $d_Vf(e) \in E^*_x$ denotes the vertical differential of the function f on E at e.

<u>Theorem 1</u>. **Ω is a symplectic form on the neighborhood U (described above) of $B \times \{0\}$ in $B \times \mathfrak{g}^*$. The G action $(b,\mu) \longmapsto (bg, Ad_g^*\mu)$ is a canonical action. The projection $B \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the momentum map for this action.**

The corresponding Poisson reduced space $E = B \times_G \mathfrak{g}^*$ has Poisson bracket given by :

$$\{f,g\}(e) = \{h^*f, h^*g\}_S(x) + \langle e, F(X_{h*f}, X_{h*g})(x)\rangle + \langle e, [d_v f, d_v g]\rangle$$
$$+ \langle e, F(Z_{h*f}, X_{h*g})(x)\rangle \qquad \text{[PB.2]}$$

The Hamiltonian vector field corresponding to the function π^*F, F a function on S is

$$X_{\pi * F}(e) = h_e J_e(dF) = h_e X_F + h_e Z_F(e) .$$

The symplectic leaf of E through $e = [b,\mu]$ is the space $B \times_G \mathcal{O}$ ($\mathcal{O}$ is the co-adjoint orbit through μ), with the symplectic structure given to it by Sternberg. The symplectic form on this leaf at e with respect to the connection-induced identification $T_e(B \times_G \mathcal{O}) \simeq T_x S \oplus T_\mu \mathcal{O}$ is $\omega_e \oplus \omega_{\mathcal{O}}$ where $\omega_{\mathcal{O}}$ is the + orbit symplectic form on $\mathcal{O}$.

Before we prove this theorem, we make several important remarks, backed up by theorems. The first four remarks have to do with the **cross-curvature term** :

$$\langle e, F(Z_{h*f}, X_{h*g})(x)\rangle.$$

<u>Remark 1.</u> Using the definition of Z_f, we note that the cross-curvature term satisfies

$$\langle e, F(Z_{h*f}, X_{h*g})(x)\rangle = \omega_e(Z_{h*f}, Z_{h*g})(x).$$

This substitution makes the brackets [PB.2] manifestly skew-symmetric.

The brackets [PB.2] can also be rewritten

$$\{f,g\}(e) = \{h^*f, h^*g\}_e(x) + \langle e, [d_v f, d_v g]\rangle \qquad \text{[PB.2*]}$$

where $\{F, G\}_e(x)$ is defined to be $\omega_e(x)(J_e(dF), J_e(dG))$, for F and G functions in a neighborhood of x. To see this we compute

$$\omega_e(J_e(dF), J_e(dG)) = \omega_e(J_0(dF)+Z_F, J_0(dG)+Z_G)$$

$$= \omega(J_0(dF), J_0(dG)) - \langle e, F(J_0(dF), J_0(dG))\rangle$$
$$+ \omega_e(J_0(dF), Z_G) + \omega_e(Z_F, J_0(dG)) + \omega_e(Z_F, Z_G)$$
$$= \{F,G\}_S(x) - \langle e,F(X_F, X_G)\rangle + \langle e,F(X_F, X_G)\rangle + \langle e,F(X_F, X_G)\rangle + \langle e,F(Z_F, X_G)\rangle$$
$$= \{F,G\}_S(x) + \langle e,F(X_F, X_G)\rangle + \langle e,F(Z_F, X_G)\rangle$$

Now, set $F = h^*f$, $G = h^*g$ and compare with [PB.2].

Since $\{h^*f,\ h^*g\}_e(x) = dh^*f(J_e dh^*g)$, we see that the Poisson bracket blows up precisely at the points e where ω_e becomes degenerate, i.e. on the complement of $p(U)$.

Remark 2. If the cross-curvature term is zero, then the bracket [PB.2] reduces to the intrinsic bracket stated at the end of the previous section. The relevant fact which makes the cross-curvature term disappear here is that kerF is a co-isotropic distribution on S. That is, for each $x \in S$, $\ker F_x \subset T_sS$ contains a Lagrangian subspace of T_sS. (When we say "kerF", we are thinking of F as a vector bundle endomorphism, TS ⟶ Hom(TS,Ad(B)).) We use the word 'distribution' in the generalized sense: the rank of $\ker F_x$ is allowed to change as x varies.

Theorem 2. The cross-curvature term $\langle e,F(Z_{h^*f}, X_{h^*g})(x)\rangle$ is zero for all f,g, and e if and only if kerF is a co-isotropic distribution on S. If this is the case then Ω is

globally a symplectic form on $B\times\mathfrak{g}^*$.

We will give the proof of this theorem below, following the proof of Theorem 1.

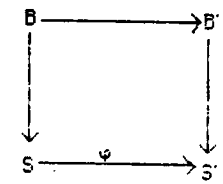
Corollary. Let S = T*X and B be the pull-back bundle of a principal bundle over X. Then the intrinsic bracket formula [PB.1] of §1.2 is valid.

Proof of the Corollary. According to Theorem 2, we need only prove that kerF is a co-isotropic distribution. The Poisson structure of the previous section is obtained by the construction of this section if we take S = T*X, A to be the pull back of a connection A_* over X. Then $F = \tau_Q^*F_*$ where τ_Q: T*X→X is the cotangent projection and F_* is the curvature of A_*. [In the previous section, the present A went by the name of $A^\#$, and the present F by the name of $F^\#$. The present A_* was called A in the previous section, and the present F_* was called F.] Hence $\ker F = \{Y \in TT^*Q: F_*(T\tau_Q Y, \cdot\,) = 0\} \subset \ker T\tau_Q$. In coordinates, kerF contains all the $\partial/\partial p$'s, and perhaps some of the $\partial/\partial q$'s (namely, the $\partial/\partial q$'s spanning $\ker F_*$) and hence is co-isotropic. ●

Remark 3. There is a local converse to the fact that the cross-curvature term is zero in the Yang-Mills case. From the Bianchi identity one sees that kerF is an involutive distribution. If it is also co-isotropic, then locally it contains a polarization (involutive Lagrangian distribution) Λ and locally $Q = S/\Lambda$ is a manifold, with $S \simeq T^*Q$, where the fibres $T_q{}^*Q$ correspond to the leaves of Λ. Since the curvature is zero along these leaves, A is the pullback of a connection A_* for the principal bundle $B_* \to Q$, where B_* is the restriction of B to the zero section of $S \simeq T^*Q$. Summarizing:

Theorem 3. **If the cross-curvature term is zero then locally $S \simeq T^*Q$ and the connection A on B is the pullback of a connection on $B|_{\text{zero-section}} \to Q$. In other words, locally the situation is that of the previous section.**

Note that the existence of such a connection, that is, a connection whose curvature has co-isotropic kernel, is a "symplectic bundle" invariant. By this we mean that if $B \to S$ admits such a connection and



is a bundle isomorphism with φ symplectic, then $B' \to S'$ admits such a connection.

Remark 4. Recall that a **polarization** of S is a Lagrangian distribution. To say that kerF is co-isotropic is equivalent to saying that it contains a polarization. Complex polarizations are important in geometric quantization and may have consequences to the work here. As evidence for this, we cite the fact that on a Kahler manifold the $\partial/\partial z^\alpha$ span a complex polarization on which the curvature vanishes.

Work in this direction would probably involve complex Poisson brackets. In this regard we take note of the following curiosity. Consider the standard Poisson bracket on $\mathbb{R}^2 \simeq \mathbb{C}$. Extend the bracket to complex functions by complex linearity. Then one finds as a consequence of the Cauchy-Riemann differential equations (or more picturesquely, by thinking of z and $\bar{z}$ as canonically conjugate variables) that the bracket of two holomorphic functions is always zero. This fact remains true if $\mathbb{R}^2$ is replaced by $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. The calculation is almost identical to the $n = 1$ calculation.

Remark 5. As suggested by Weinstein ([1978], and in conversation) the form Ω of [1.2.1] on $B \times \mathfrak{g}^*$ is the pullback of a form on T^*B. The form on T^*B is $\tilde{\Omega} = \omega_B + (pr)^*\omega_S$ where $\omega_B = -d\theta_B$ is the canonical symplectic form and pr is the projection $T^*B \to B \to S$. Embed $B \times \mathfrak{g}^*$ into T^*B by the composition

$$B \times \mathfrak{g}^* \xrightarrow{O \times id.} B^\# \times \mathfrak{g}^* \xrightarrow{\Phi(A)} T^*B$$

where $B^\# \subset T^*B$, and $\Phi(A)$ are as in §1.1, and O is the zero section $b \mapsto O_b \in B_b^\# \subset T_b^*B$. The pullback of $\tilde{\Omega}$ by this embedding is Ω. The embedding is also G-equivariant, so it induces an embedding of $Ad^*(B)$ into T^*B/G. This is not a Poisson embedding, but its image inherits a Poisson structure (with possible singularities) from that of T^*B/G in the same way that a submanifold which is transverse to a symplectic leaf in a general Poisson manifold inherits a Poisson structure (called the "transverse" Poisson structure, see Weinstein [1984, prop. 1.4]). Our embedding Φ is an isomorphism between $Ad^*(B)$ with the Poisson structure of Theorem 1, and its image in T^*B/G with its inherited Poisson structure.

Proof of theorem 1

The connection A defines a decomposition:

$$T_{(b,\mu)}(B \times \mathfrak{g}^*) \simeq T_xS \times \mathfrak{g} \times \mathfrak{g}^*, \text{ given by } (Y,\alpha) \mapsto (T\pi_b Y, A_b Y, \alpha) \qquad [1.2.3]$$

with the inverse isomorphism given by

$$(v,\gamma,\alpha) \mapsto (h_b v + \sigma_b \gamma, \alpha)$$

where $h_b: T_xS \to T_bB$ denotes horizontal lift and $\sigma_b: \mathfrak{g} \to T_bB$ denotes the infinitesimal generator operator. Now

$$d\langle\mu,A\rangle = \langle\mu,dA\rangle + \langle d\mu \wedge A\rangle$$

$$= \langle\mu,F\rangle - \langle\mu,[A,A]\rangle + \langle d\mu \wedge A\rangle$$

where the notation is self-explanatory. Relative to our decomposition, we have:

$$\Omega((v_1,\gamma_1,\alpha_1),(v_2,\gamma_2,\alpha_2)) = \omega(v_1,v_2) - \langle\mu,F(hv_1,hv_2)\rangle + \langle\mu,[\gamma_1,\gamma_2]\rangle$$
$$- \langle\alpha_1,\gamma_2\rangle + \langle\alpha_2,\gamma_1\rangle$$

$$= \omega(v_1,v_2) - \langle e,F(v_1,v_2)\rangle + \langle\mu,[\gamma_1,\gamma_2]\rangle - \langle\alpha_1,\gamma_2\rangle + \langle\alpha_2,\gamma_1\rangle$$

$$= \omega_e(v_1,v_2) + \langle \mu, [\gamma_1,\gamma_2]\rangle - \langle \alpha_1,\gamma_2\rangle + \langle \alpha_2,\gamma_1\rangle \qquad [1.2.4] \; .$$

Note that this two-form can be written $\omega_e \oplus \sigma(\mu)$ on $T_xS \oplus (\mathfrak{g} \times \mathfrak{g}^*)$ where $\sigma(\mu)$ is given by the last three terms of [1.2.4]. In coordinates $\sigma(\mu)$ is of the form

$$\begin{matrix} 0 & I \\ -I & \lambda \end{matrix}$$

which is always invertible. Thus $\Omega(b,\mu)$ is nondegenerate if and only if ω_e is. **This proves that Ω is symplectic precisely on U**.

The decomposition dual to [1.2.3] is

$$T_{(b,\mu)}^*(B \times g^*) \cong T_x^*S \times \mathfrak{g}^* \times \mathfrak{g}, \quad \text{by } (\beta,\gamma) \mapsto (h_b^*\beta, \sigma_b^*\beta, \gamma) \qquad [1.2.3^*]$$

with inverse

$$(\xi,\alpha,\gamma) \mapsto (T\pi_b^*\xi + A_b^*\alpha, \gamma).$$

It will be helpful to calculate the vectors Ω-dual to each of the three types of covectors in this decomposition.

A covector in the first factor has the form $(\beta,0,0)$ with $\beta \in T_x^*S$. Now

$$(\beta,0,0)\cdot(v_2,\gamma_2,\alpha_2) = \beta\cdot v_2 = \omega_e(J_e(\beta),v_1) = \Omega((J_e(\beta),0,0)),(v_2,\gamma_2,\alpha_2))$$

which shows that the Ω-dual of $(\beta,0,0)$ is $(J_e(\beta),0,0) = (J_0(\beta)+Z_e(\beta),0,0)$.

Under our decompositions, $(\beta,0,0)$ corresponds to $(\pi^*\beta,0) = p^*\pi^*\beta \in T_b^*B\times\mathfrak{g}$ and $(J_0(\beta)+Z_e(\beta), 0,0)$ to $(h_bJ_0(\beta)+h_bZ_e(\beta),0) \in T_bB\times\mathfrak{g}^*$. Set $\beta = dF$ where F is a function on S. Then the Hamiltonian vector field on $B \times \mathfrak{g}^*$ corresponding to $p^*\pi^*F$ is $(h_bX_F+h_bZ_F(e), 0)$. Since p is a Poisson map, and since the horizontal lift $h_e: T_xS \to T_eE$ is equal to $T_{[b,\mu]}p_*(h_b \oplus 0)$, **we see that $X_{\pi^*F}(e) = h_eX_F + h_eZ_F(e)$ as stated in the theorem**.

A covector in the second factor has the form $(0,\alpha,0)$ where $\alpha \in \mathfrak{g}^*$. According to formula [1.2.4]

$$\Omega((0,0,-\alpha),(v_2,\gamma_2,\alpha_2)) = 0 + 0 + \langle\alpha,\gamma_2\rangle + 0$$

so that the Ω-dual to $(0,\alpha,0)$ is $(0,0,-\alpha)$.

Finally, a covector in the third factor has the form $(0,0,\gamma)$, $\gamma \in \mathfrak{g}$. Its dual vector is $(0,\gamma,ad_\gamma^*\mu)$ since

$(0,0,\gamma)(v_2,\gamma_2,\alpha_2) = \langle\alpha_2,\gamma\rangle$

$$= \langle\mu,[\gamma,\gamma_2]\rangle - \langle\mu,[\gamma,\gamma_2]\rangle + \langle\alpha_2,\gamma\rangle$$

$$= \langle\mu,[\gamma,\gamma_2]\rangle - \langle ad_\gamma{}^*\mu,\gamma_2\rangle + \langle\alpha_2,\gamma\rangle$$

$$= \Omega((0,\gamma,ad_\gamma{}^*\mu),(v_2,\gamma_2,\alpha_2)) .$$

Now $(0,\gamma,ad_\gamma{}^*\mu)$ corresponds under our decomposition to the infinitesimal generator $\gamma_{B\times\mathfrak{g}^*}$ under the 'diagonal' right G action. Thus the preceding calculation says that this infinitesimal generator is the Ω-Hamiltonian vector field for the Hamiltonian $\langle\pi_2,\gamma\rangle$ where π_2 is the projection $B\times\mathfrak{g}^*\to\mathfrak{g}^*$. **This proves the claim that π_2 is the momentum map for the diagonal G action.**

From the general theory of Poisson reduction [see appendix], it follows that the symplectic leaves of E are the submanifolds $J^{-1}(\Theta)/G = B\times_G\Theta$. The symplectic form on a leaf is obtained by adding $\pi_2{}^*\omega_\Theta = J^*\omega_\Theta$, where ω_Θ is the +orbit symplectic form on Θ, to Ω restricted to $B\times\Theta = J^{-1}(\Theta)$ and pushing the resulting sum down to $J^{-1}(\Theta)/G$. So, the form on $J^{-1}(\Theta) = B\times\Theta$ which one pushes down is $\pi^*\omega_S - d\langle\mu,A\rangle + \pi_2{}^*\omega_\Theta$. This is precisely the form which Sternberg [1977] pushed down to get his pre-symplectic structure on $B\times_G\Theta$. **This proves that the symplectic leaves of E are the spaces $B\times_G\Theta$ with the symplectic structure given to them by Sternberg.**

We now show that the symplectic form on the leaf $B\times_G\Theta$ is given by our formula $\omega_e\oplus\omega_\Theta$. According to the above paragraph it suffices to show that $p^*\omega_e\oplus\omega_\Theta = \Omega + \pi_2{}^*\omega_\Theta$. Relative to the decomposition [1.2.3] and the connection-induced decomposition $T_eE \simeq T_xS\times\mathfrak{g}^*$ the differential $T_{(b,\mu)}p$ is given by $(v,\gamma,\alpha)\mapsto(v,\alpha-ad_\gamma{}^*\mu)$. Let $X_1, X_2 \in T_{(b,\mu)}B\times\Theta$. With respect to our decomposition, $X_i = (v_i,\gamma_i,\alpha_i)$, for $i = 1,2$, where $\alpha_i = ad^*(\beta_i)\mu$ (since $\alpha_i \in T_\mu\Theta$). Set $\overline{X}_i = Tp\cdot X_i$ so that $\overline{X}_i = (v_i, ad(\beta_i-\gamma_i)^*\mu)$. Then

$$p^*(\omega_e\oplus\omega_\Theta)(X_1,X_2) = \omega_e\oplus\omega_\Theta(\overline{X}_1,\overline{X}_2)$$

$$= \omega_e(v_1,v_2) + \omega_\Theta(ad(\beta_1-\gamma_1)^*\mu, ad(\beta_2-\gamma_2)^*\mu)$$

$$= \omega_e(v_1,v_2) + \langle\mu,[\beta_1-\gamma_1,\beta_2-\gamma_2]\rangle .$$

where in the last equality we used the definition of ω_Θ. From formula (1.2.4) we have

$$\Omega(X_1,X_2) + \pi_2{}^*\omega_\Theta(X_1,X_2) = \omega_e(v_1,v_2) + \langle\mu,[\gamma_1,\gamma_2]\rangle - \langle\alpha_1,\gamma_2\rangle$$

$$+ \langle\alpha_2,\gamma_1\rangle + \omega_\Theta(\alpha_1,\alpha_2)$$

$$= \omega_e(v_1,v_2) + \langle\mu,[\gamma_1,\gamma_2]\rangle - \langle\mu,[\beta_1,\gamma_2]\rangle + \langle\mu,[\beta_2,\gamma_1]\rangle$$

$$+ \langle\mu,[\beta_1,\beta_2]\rangle$$

$= \omega_e(v_1, v_2) + \langle \mu, [\beta_1-\gamma_1, \beta_2-\gamma_2]\rangle$.

where in the second equality we used $\alpha_i = ad^*(\beta_i)\mu$. **This proves the final claim of the Theorem: the symplectic form on the leaf $B \times_G \mathcal{O}$ is $\omega_e \oplus \omega_{\mathcal{O}}$.**

We are now in a position to compute brackets on E. Call a function horizontal at e if its differential annihilates the space of vertical vectors at e, and vertical if its differential annihilates the space of horizontal vectors at e. We will only calculate brackets for functions which are either horizontal or are vertical at the point $e \in E$ in question. The differentials of such functions clearly span T^*_eE. Thus if we show that formula [PB.2] holds for the three possible combinations: {horizontal,horizontal}, {horizontal,vertical}, and {vertical, vertical}, then we have proved the validity of that formula (since both sides of this formula depend only on the differentials of the functions at e, and in a bilinear skew-symmetric way).

{horizontal,horizontal}. The horizontal functions at e are all of the form π^*f, f a function on S (i.e. any horizontal covector at e can be written $\pi^*df(x)$). We have

$$\{\pi^*f, \pi^*g\}(e) = \{p^*\pi^*f, p^*\pi^*g\}_{B\times\mathfrak{g}^*}(b,\mu)$$

Now

$$\begin{aligned}\{p^*\pi^*f, p^*\pi^*g\}_{B\times\mathfrak{g}^*}(b,\mu) &= \Omega(\,(h_bJ_e(df),0),\ (h_bJ_e(dg),0)\,)\\ &= \omega_e(J_e(df),J_e(dg))\\ &= \{f,g\}_e(x)\end{aligned}$$

Note in remark 1, we showed that $\{f,g\}_e(x) = \{f,g\}_S(x) + \langle e,F(X_f, X_g)(x)\rangle + \langle e,F(Z_f, X_g)(x)\rangle$.

vertical terms. To calculate brackets with vertical terms it is helpful to have a nice representation of these functions. **Any vertical function can be written as pairing with a section ψ of E^* which is flat at x.** That such a function is vertical easily shown:

$$d\langle e,\psi\rangle = \langle e,D\psi\cdot T\pi\rangle + \langle de,\psi\rangle = \langle de,\psi\rangle$$

where de denotes the vertical projection onto E_x. That all vertical differentials arise in this way can be seen by a dimension count. As usual, ψ can be thought of as an equivariant function $B\to\mathfrak{g}$ which will also be denoted by ψ. The pull-back by p of the function $e \mapsto \langle e,\psi(x)\rangle$ is the function

$$(b,\mu) \mapsto \langle \mu,\psi(b)\rangle$$

In order to apply our formula for Ω we must compute the differential of this function relative to our decomposition, . Its differential with respect to the μ slot is clearly $\psi(b)$. Since $\langle\mu,\psi(bg)\rangle = \langle\mu,Ad_{g^{-1}}\psi(b)\rangle$, its differential with respect to a vertical vector $\sigma_b\gamma$ is $\langle\mu,[\psi(b),\gamma]\rangle = \langle ad_{\psi(b)}{}^*\mu,\gamma\rangle$. Finally, since ψ is flat at x, $d\psi$ is zero on any horizontal vectors at b. Summarizing, we have, relative to our decomposition [1.2.3*] of $T^*(B\times\mathfrak{g}^*)$,

$$dp^*\psi(b,\mu) = (0,\ ad_{\psi(b)}{}^*\mu,\ \psi(b))$$

where, by abuse of notation, we let ψ also denote the function on E which it defines via pairing. Thus the Hamiltonian vector on $B\times\mathfrak{g}^*$ corresponding to $p^*\psi$ is

$$X_{p^*\psi}(b,\mu) = \Omega\text{-dual of}[\ (0,\ Ad_{\psi(b)}{}^*\mu,\ 0)\ +\ (0,\ 0,\ \psi(b))\ ]$$
$$= (0,\ 0,-Ad_{\psi(b)}{}^*\mu,\ ) + (0,\psi(b),\ Ad_{\psi(b)}{}^*\mu,)$$
$$= (0,\psi(b),\ 0)$$

<u>[horizontal,vertical]</u>:

$$\{\pi^*f,\ \psi\}_E = \{p^*\pi^*f,\ p^*\psi\}_{B\times\mathfrak{g}^*}$$
$$= \Omega((X_f + Z_f,0,0),\ (0,\psi(b),\ 0))$$
$$= 0$$

<u>[horizontal, horizontal]</u>:

$$\{\psi,\varphi\}_E = \{p^*\psi,\ p^*\varphi\}_{B\times g^*}$$
$$= \Omega((0,\psi(b),\ 0),(0,\varphi(b),\ 0))$$
$$= \langle\mu,[\psi(b),\varphi(b)]\rangle$$

To see that these three expressions agree with the left hand side of [PB.2], we simply note that horizontal functions f satisfy $f = h^*\pi^*f$ since $\pi\circ h$ = identity, and that vertical functions ψ satisfy $\psi = d_V\psi$ and $dh^*\psi = 0$ (since $D\psi(e) = 0$). This completes the proof of Theorem 1.●

For completeness, we record the Hamiltonian vector field X_ψ on E corresponding to the function $\psi \in \Gamma(E^*)$, thought of as a fibre-linear function on E. We use the representation of vertical functions used in the proof of the theorem. We have $p^*\psi(b,\mu) = \langle\mu,\psi(b)\rangle$ [taking advantage of the same abuses of notation that we did in the proof of the theorem.] Then we calculate that $dp^*\psi(b,\mu) = (\langle\mu,D\psi\rangle,\ ad_{\psi(b)}{}^*\mu,\ \psi(b))$. So the Hamiltonian vector field for $p^*\psi$ at (b,μ) is

$$X_{p^*\psi} = \Omega\text{-dual of } [(\langle\mu,D\psi\rangle,\ 0\ ,\ 0) + (0,\ ad_{\psi(b)}{}^*\mu\ ,\ \psi(b))]$$
$$= (J_e[\ \langle\mu,D\psi\rangle],\ 0\ ,\ 0\ ) + (\ 0\ ,\ \psi(b)\ ,\ 0\ ).$$

Pushing X down to E by Tp yields:

$$X_\psi(e) = h_e J_e(\langle e, D\psi\rangle) + \psi_E(e) \qquad [1.2.5]$$

Here ψ_E is the infinitesimal generator corresponding to $\psi \in \Gamma(E^*) = \Gamma(Ad(B))$, the Lie algebra of the group of automorphisms of B. This group acts on the left on E^* by vector bundle automorphisms.

Proof of Theorem2:

Any differential df at $x \in S$ equals $dh^*\pi^*f(x)$. Any vector at x is of the form X_f for some f. As e runs over E_x, μ runs through $\mathfrak{g}^*$, where $e = [b,\mu]$, and $\langle e, F(Z_f, X_g)(x)\rangle = \langle \mu, F(hZ_f, hX_g)(b)\rangle$. Therefore, the cross-curvature term is zero if and only if $Z_f \in$ kerF for all functions f on S. We show that $\{Z_f$: f a function on S$\} = (\ker F)^{\omega\text{-perp}}$, whenever ω_e is nondegenerate.It follows that kerF is ω-coisotropic ($(\ker F)^{\omega\text{-perp}} \subset \ker F$) if and only if $Z_f \in$ kerF for all functions f on S.

We start by recalling some notation. If Λ is a subspace of T_xS, then Λ^0 denotes the annihilator of Λ, which is a subspace of $T_x{}^*S$. Let $J: T_x{}^*S \to T_xS$ ($J_e: T_x{}^*S \to T_xS$) denote the isomorphism induced by ω (ω_e). So $X_f = J(df)$, and $J_e(\langle e, F(X_f, \cdot\,)\rangle) = Z_f$. Now we record some elementary facts:

(i) $\{ \langle e, F(X_f, \cdot\,)(x)\rangle : e \in E_x$, f a function on S$\} = (\ker F)^0$.

(ii) $(\ker F)^{\omega\text{-perp}} = (\ker F)^{\omega_e\text{-perp}}$.

(iii) $J(\Lambda^0) = \Lambda^{\omega\text{-perp}}$, $J_e(\Lambda^0) = \Lambda^{\omega_e\text{-perp}}$, for all subspaces Λ of T_xS.

The only one of these facts which is slightly tricky is (ii). If $Z \in$ kerF, then for all v, $\omega(Z,v) = \omega_e(Z,v)$. Hence for fixed v, if one side is zero for all $Z \in$ kerF, then so is the other side.

Using (i), the definition of Z_f, and (iii), we see that $\{Z_f$: f a function on S$\} = (\ker F)^{\omega_e\text{-perp}}$. Using (ii), we have $\{Z_f$: f a function on S$\} = (\ker F)^{\omega\text{-perp}}$. therefore $\{Z_f$: f a function on S$\} \subset$ (kerF) iff $(\ker F)^{\omega\text{-perp}} \subset \ker F$.

Finally we show that if kerF is co-isotropic, then ω_e is always nondegenerate. Pick a Lagrangian subspace contained in kerF and a Darboux basis for T_xS, the last half of whose elements span this subspace. Then, relative to this basis:

$$\omega = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$$

and $\langle e,F\rangle$ has the form

$$\begin{matrix} F & 0 \\ 0 & 0 \end{matrix} \quad ,$$

so that ω_e has the form

$$\begin{matrix} -F & 1 \\ -1 & 0 \end{matrix}$$

which is always invertible. Its inverse is

$$\begin{matrix} 0 & -1 \\ 1 & -F \end{matrix} .$$

This demonstrates that ω_e is nondegenerate and completes the proof of Theorem 2. •

§1.3 The Normal Bundle to a Co-adjoint Orbit and Poisson Fibre Bundles.

Introduction.

In this section we investigate Poisson structures on the normal bundle N(L) to the symplectic leaf L of a Poisson manifold P. We will be concerned mainly with the case L = $\mathcal{O}$ a reductive (defined below) co-adjoint orbit in P = $\mathfrak{g}^*$. (Every co-adjoint orbit is reductive if G admits a bi-invariant Riemannian metric, and in particular if G is compact.) These structures will give examples where the cross-curvature term of the previous section is not zero, and also where the Poisson structure blows up. The reader is urged to look ahead near the end of this section where the example N(S^2) for G=SO(3) and N(principal orbit) for G=SU(3) are presented.

Returning to the general case, the fibres of N = N(L) inherit a linear Poisson structure from P. On the other hand, one can use an exponential map N⟶P to pull back the Poisson structure on P thus obtaining a Poisson structure on N in a neighborhood of its zero section. This Poisson structure induces the transverse Poisson structure (see Weinstein [1984] for a definition) on the fibers of N. If these two Poisson structures on the fibers of N are equal we will call the exponential map a "simultaneous linearization" of L and will say that L is "simultaneously linearizable". Reductive orbits have a natural exponential map N($\mathcal{O}$)⟶$\mathfrak{g}^*$. Molino [1984] noted that this map is a simultaneous linearization of $\mathcal{O}$.

A simultaneous linearization of N(L) provides a way of putting together the Lie-Poisson structure on the fibres of N and the symplectic structure on the base L of N in order to obtain a Poisson structure on N. In the previous section, we saw how to do a similar type of splicing: we used a connection to put a Poisson structure on Ad*(B) where B is a principal bundle over the symplectic manifold L. Now $B=G \rightarrow \Theta = G/G_\mu$ is a principal G_μ-bundle where G_μ is the isotropy group of an element μ of Θ. In the case where Θ is a reductive orbit there is a natural G-invariant connection on this principal bundle. (In fact, the existence of such a connection is equivalent to Θ being reductive.) It also turns out that in the reductive case the associated vector bundle Ad*(B) to B=G is identifiable with N(Θ). So, if we apply the construction of the previous section to this situation, we obtain another Poisson structure on N(Θ) in addition to Molino's. The basic result of this section [Theorem 1 below] is that these two Poisson structure agree.

This result can be proven as a corollary of the work of Guillemin and Sternberg [1985], as we show in remark 2 below [p.51-52]. However Guillemin and Sternberg do not state or prove the result. The proof which we give is self-contained.

As a corollary to Theorem 1, we get a singularity theorem, which says that the set of points where Molino's Poisson structure on N(Θ) blows up equals the set of critical points of the exponential map. (This theorem is not obvious, since for a general simultaneous linearization of L it is not true that critical points of the exponential map are singularities of the Poisson structure on N(L).) Nice

examples of this are provided by Molino's Poisson structure on N(S^2), and more generally on N(Θ), Θ a regular co-adjoint orbit for G a compact group.

The lay-out of this section is as follows. First we discuss the fiber Poisson structure on N(L) and put it into the more general context of what we call "Lie-Poisson vector bundles". We then review the notion of linearizing the Poisson structure at $x \in L$ as discussed by Weinstein [1984]. We will prove that if P is linearizable at x, then L is simultaneously linearizable (the converse is obvious). This proof was shown to me by Weinstein. Thus linearizing is equivalent to simultaneously linearizing. Even so, we feel that the concept of simultaneously linearization is useful, if for no other reason than that it generate interesting examples of Poisson manifolds and of singularities of Poisson structures. After this we will discuss structure groups for N(L) and for Lie-Poisson vector bundles in general. After these preliminaries, we come to the body of this section, which is the investigation of the case L= Θ a reductive co-adjoint orbit. We define the two Poisson structures on N(Θ), and prove in Theorem 1 that these two structures are equal. Finally, we will prove the Singularity Theorem concerning how the Poisson structure on N(Θ) blows up, and illustrate this blow-up with the examples N(S^2) and more generally N(Θ), Θ a regular co-adjoint orbit for a compact G.

<u>The Normal Bundle as a Poisson Fibre Bundle.</u>

The normal bundle has fibres $N_x = T_xP/T_xL$. So any linear function on the fibre N_x can be written $df(x)$ where f is a function on P which is constant on L. The Poisson bracket on N_x is then given by

$$\{df(x), dg(x)\}_N = (d\{f,g\}_P)(x) .$$

A submanifold of P which is transverse to L at x inherits a Poisson structure in a neighborhood of x, called the transverse Poisson structure at x. Any choice of a transverse submanifold yields a locally isomorphic Poisson structure, and the local isomorphism class of this Poisson structure is independent of which $x \in L$ is picked. Another way of characterizing the Poisson structure on N_x is that it is the linearization of the transverse Poisson structure to L at x. For more details see [Weinstein, 1984]. The linearization problem for Poisson manifolds asks: "Is N_x Poisson-isomorphic to the transverse Poisson structure at x?" If this is the case we will say that P is linearizable at x. The linearization problem is discussed in detail by Weinstein [1984]. Since the transverse structure is independent of the base point $x \in L$, we see that if P is linearizable at x, then it is linearizable at any other point y of L. **It is thus proper to speak of an entire leaf L as being linearizable or not. This leads us to suspect that if L is linearizable**

then it is in some sense "simultaneously linearizable". We have already defined a concept of simultaneous linearizability in the introduction. However it will prove useful to give another equivalent definition in order to prove this conjecture. We begin with some definitions.

A **Poisson fibre bundle** is a fibre bundle E whose fibers E_x have a Poisson structure which varies smoothly from fibre to fibre. If all the fibres are Poisson isomorphic, we will call the fibre bundle a **regular** Poisson vector bundle. If the fibre bundle is a vector bundle and if on each fibre the bracket of linear functions is linear then we call E a **Lie-Poisson vector bundle**. N(L) of the present section and Ad*(B) of the previous section are examples of regular Lie-Poisson vector bundles over symplectic base manifolds. The Lie-Poisson vector bundle structure on E = Ad*(B) is obtained by putting the Poisson structure on the fibres $E_x \simeq \mathfrak{g}^*$ which they inherit as transverse manifolds to the zero-section, which is a symplectic leaf in E. This Lie-Poisson vector bundle structure on Ad*(B) is independent of the choice of connection on B.

A **realization** of a regular Lie-Poisson vector bundle E over a symplectic base is a Poisson structure on a neighborhood in the total space E which contains the zero section and such that

(i) the zero section $S \hookrightarrow E$ is a Poisson map (in particular its image is a symplectic leaf in E)

(ii) the transverse Poisson structure on the fibres agrees with the

Poisson structure they inherit from the Lie-Poisson vector bundle structure.

Example: The Poisson structure on $Ad^*(B)$ is a realization of its Lie-Poisson vector bundle structure.

We are now in a position to state the

Simultaneous linearization problem: Is there an exponential map $N \to P$ such that the pullback of the Poisson structure on P to N is a realization of the Pvb structure on N?

If such a map exists, we will say that P is **simultaneously linearizable about** L, or simply that L is **simultaneously linearizable** and we call the resulting realization a **linearizing realization.**

(By an **exponential map**, we mean a map $\exp: N \to P$ satisfying:
(i)The composition

$$L \xrightarrow{0} N \xrightarrow{\exp} L$$

is the identity, where 0 denotes the zero section of N, and (ii): the composition

$$N_x \xrightarrow{\approx} V_{0(x)} \hookrightarrow T_{0(x)}N \xrightarrow{T_{0(x)}\exp} T_xP \to T_xP/T_xL = N_x$$

is the identity, where $V = \ker T\pi \subset TN$ denotes the subbundle of vertical vectors and the first isomorphism is the usual one between V_e and $E_{\pi(e)}$ for any vector bundle E. It follows that an exponential map is an isomorphism in a neighborhood of the zero section. The usual exponential map induced by a Riemannian metric on P is of course an exponential map in this present sense.)

If L is simultaneously linearizable then $U \subset N(L)$ is Poisson isomorphic to a tubular neighborhood of L in P. It will not in general be true that N is isomorphic to all of P.

We conjectured above that if L is a linearizable leaf then it is in fact simultaneously linearizable. We now show this. Suppose that L is linearizable. Let $E \subset P$ be a tubular neighborhood of L and $E \to L$ the corresponding projection with fibres E_x. The E_x are transverse manifolds to L. As such they inherit the transverse Poisson structure, so that E becomes a Poisson fibre bundle. The fact that L is linearizable means (after perhaps making E smaller) that for each $x \in L$ there is a Poisson isomorphism $\varphi_x: N_x \to E_x$ and which agrees to first order with the natural linear isomorphism $N_x \to T_xE_x$. The problem is then to put these isomorphisms together in a smooth way in order to form a fibre bundle map $\varphi: N \to E$. Such a φ is an exponential map, and the pull-back by φ of the Poisson structure on $E \subset P$ to N is a realization of the Lie-Poisson vector bundle structure of N. To see that φ indeed exists, consider the bundle Hom(N,E) over L whose fibre over x is the set of such Poisson maps φ_x. Our hypothetical φ is a global cross-section of this bundle. We will show that the fibers of this bundle are contractible, and hence that such a global cross-section φ exists. If φ_x, $\gamma_x \in \mathrm{Hom}_x(N,E)$ then $\varphi_x^{-1} \cdot \gamma_x$ is a Poisson automorphism of N_x whose differential at 0 is the identity. Thus the fibre of Hom(N,E) can be identified with the subgroup of all Poisson autmorphisms of $\mathfrak{g}^*$ whose differential at 0 is the identity. Now, if φ is a Poisson automorphism of $\mathfrak{g}^*$, so is φ_t, where $\varphi_t(\mu) = 1/t(\varphi(t\mu))$. As $t \to 0$, $\varphi_t \to D\varphi(0)$. This demonstrates that the group of all Poisson automorphisms of $\mathfrak{g}^*$ can be contracted onto Aut($\mathfrak{g}^*$), the group of linear Poisson automorphisms of $\mathfrak{g}^*$. Moreover, if φ is in our subgroup (i.e. $D\varphi(0) = \mathrm{id.}$), then so is φ_t. But then $\varphi_0 = D\varphi(0) = \mathrm{id.}$, so we have shown that our fibre is contractible to a point.

Structure Groups and an Example of a Lie-Poisson Vector Bundle which is not the Normal Bundle to a Symplectic Leaf.

The structure group of a general Lie-Poisson vector bundle can be reduced from Gl($\mathfrak{g}^*$) to Aut($\mathfrak{g}^*$), the group of linear Poisson automorphisms of $\mathfrak{g}^*$. The structure group of N(L) can be reduced further in the sense that one can put a finer topology on Aut($\mathfrak{g}^*$) so that there are fewer allowable transition functions. The standard

topology on Aut($\mathfrak{g}^*$) is the one it inherits as a submanifold of Gl($\mathfrak{g}^*$). To obtain the finer topology, consider the fibration

In($\mathfrak{g}^*$)$\rightarrow$ Aut($\mathfrak{g}^*$)$\rightarrow$Out($\mathfrak{g}^*$) where Out($\mathfrak{g}^*$) = Aut($\mathfrak{g}^*$)/In($\mathfrak{g}^*$) and where In($\mathfrak{g}^*$) is the (normal) subgroup of inner automorphisms of Aut($\mathfrak{g}^*$). The group Out($\mathfrak{g}^*$) of outer automorphisms shuffles around symplectic leaves of $\mathfrak{g}^*$. Since flows generated by Hamiltonians cannot shuffle leaves around, and since Hamiltonians on P can be used to generate the transition functions for N(L) the transition functions for N(L) can be taken to have locally constant Out($\mathfrak{g}^*$) components. For more details, see Dazord [1984]. To guarantee that these components of the transition functions are locally constant, we put the discrete topology on Out($\mathfrak{g}^*$). The topology of In($\mathfrak{g}^*$) remains the same. That is to say, the finer topolgy which we take on Aut($\mathfrak{g}^*$) is the one which makes the fibration isomorphic Out($\mathfrak{g}^*$)$_{\mathrm{discrete}}$x In($\mathfrak{g}^*$). If $\mathfrak{g}$ is the semisimple Lie algebra for a compact group then Out($\mathfrak{g}^*$) is finite, so that the two topologies on Aut($\mathfrak{g}^*$) are the same. At the opposite extreme, if $\mathfrak{g}$ is Abelian, then Aut($\mathfrak{g}^*$) = Out($\mathfrak{g}^*$) = Gl($\mathfrak{g}^*$), and is to be taken with the standard topology for arbitrary Lie-Poisson vector bundles, but with the discrete topology for normal bundle to symplectic leaves.

As an example, consider the tangent bundle to the sphere, TS^2. This has the structure of as a Lie-Poisson vector bundle over the symplectic manifold S^2 where the fibres $\mathbb{R}^2$ have the trivial Poisson structure. This Lie-Poisson vector bundle cannot be the normal bundle of S^2 as the symplectic leaf in some Poisson manifold. For, if it were, we could reduce the structure group to that of $Gl(\mathbb{R}^2)$ with the discrete topology, and any bundle with a discrete structure group over a simply connected manifold is a trivial bundle, which TS^2 is not.

<u>Normal bundle to a co-adjoint orbit.</u>

From now on we will restrict our attention to the case $L = \mathcal{O}$, a co-adjoint orbit in $P = \mathfrak{g}^*$. It is known that there are Lie algebras, for example $\mathfrak{sl}(3,\mathbb{C})$, whose duals contains nonlinearizable co-adjoint orbits. Weinstein [1984] proved that **reductive** co-adjoint orbits (defined below) are linearizable. Molino [1984] then constructed a simultaneously linearization for reductive co-adjoint orbits.

Let $\mu \in \mathcal{O}$ and $\mathfrak{g}_\mu$ denote the isotropy subgroup of μ. Then μ is called **reductive** if there is a vector space splitting

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m}_\mu \qquad [1.3.1]$$

of $\mathfrak{g}$ satisfying

$$[\mathfrak{g}_\mu, \mathfrak{m}_\mu] \subset \mathfrak{m}_\mu \,. \qquad [1.3.2]$$

If G has a bi-invariant positive definite inner product, then every element is automatically reductive, with $\mathfrak{m}_\mu = \mathfrak{g}_\mu^\perp$. If μ is reductive, then every element $g\mu = Ad_{g^{-1}}^*\mu$ of $\mathcal{O}$ is also reductive with corresponding splitting

$$\mathfrak{g} = \mathfrak{g}_{g\mu} \oplus \mathfrak{m}_{g\mu}$$

where

$$\mathfrak{g}_{g\mu} = Ad_{g^{-1}}\mathfrak{g}_\mu \;; \quad \mathfrak{m}_{g\mu} = Ad_{g^{-1}}\mathfrak{m}_\mu \,.$$

We thus speak of the entire orbit $\mathcal{O}$ as being reductive.

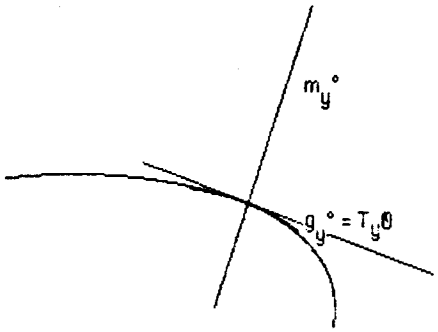
Using the reductive splitting [1.3.1], Molino [1984] constructed an exponential map $N = N(\mathcal{O}) \longrightarrow \mathfrak{g}^*$, which simultaneously linearized $\mathcal{O}$. The reductive splitting also induces a natural connection on $N \longrightarrow \mathcal{O}$, so using the construction in §1.2, we get another Poisson structure on N. We will show that these two Poisson structures on N are actually the same.

Molino's exponential map is

$$(y,\alpha) \longmapsto y + \alpha$$

where we have identified the normal bundle N with pairs

$$(y,\alpha)\,,\; y \in \mathcal{O}\;,\; \alpha \in \mathfrak{m}_y^* = \text{annihilator of } \mathfrak{m}_y.$$



Note that if the reductive splitting is induced by a bi-invariant inner product on $\mathfrak{g}$ as described above, then Molino's exponential map is the standard exponential map for the normal bundle of the submanifold Θ in the Euclidean space $\mathfrak{g}^*$. Molino's Poisson structure on N is the pull-back of the Lie-Poisson structure on $\mathfrak{g}^*$ by this exponential map.

To describe the connection-induced Poisson structure on N it is helpful to begin by working in a more general setting. A vector space splitting

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

of the Lie algebra $\mathfrak{g}$ is called **reductive**, if its factors satisfy

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \qquad [1.3.1a]$$

$$[\mathfrak{k},\mathfrak{m}] \subset \mathfrak{m} \qquad [1.3.1b]$$

that is, $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{m}$ is a sub-vector space which is invariant under the $ad_{\mathfrak{k}}$ action. (If, in addition, the splitting satisfies $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$ then the corresponding homogeneous space is called a symmetric space.) Think of $\mathfrak{m}$ as the horizontal space at $e \in G$ for a left G-invariant connection on the right principal K bundle $G \to G/K$ where K's Lie algebra is $\mathfrak{k}$. The horizontal space at $g \in G$ is then

$$H_g = TL_g\mathfrak{m}$$

Condition [1.3.1b] is equivalent to the invariance property

$$TR_kH_g = H_{gk}, \text{ for all } k \in K.$$

which connection-defined horizontal subspaces must satisfy.

The corresponding horizontal and vertical projections on TG will be denoted h and v. In particular at e, $T_eG = \mathfrak{g}$, and we have the projections

$$h:\mathfrak{g} \to \mathfrak{m} \qquad , \qquad v:\mathfrak{g} \to \mathfrak{k}$$

and the dual projections

$$h^*: \mathfrak{m}^* \to \mathfrak{g}^* \quad , \quad v^*: \mathfrak{k}^* \to \mathfrak{g}^*$$

which satisfy

$\mathrm{im}(h^*) = \mathfrak{k}^\circ$, the annihilator of $\mathfrak{k}$; $\mathrm{im}(v^*) = \mathfrak{m}^\circ$.

The $\mathfrak{k}$-valued connection one-form A corresponding to this choice of horizontal is easily described in terms of the Maurer-Cartan $\mathfrak{g}$-valued one-form φ on G:

$$\varphi_g(Y_g) = TL_g^{-1} Y_g \in \mathfrak{g} \; , Y_g \in T_g G.$$

Then

$$A_g = v \cdot \varphi_g : T_g G \to \mathfrak{k}.$$

The curvature of the connection is left G-invariant and is given at the identity by

$$F(e)(\gamma, \xi) = -v \cdot [\gamma, \xi] \; , \gamma, \xi \in \mathfrak{m} \qquad [1.3.3].$$

In our case, $\mathfrak{k} = \mathfrak{g}_\mu$, $\mathfrak{m} = \mathfrak{m}_\mu$, $K = G_\mu$ the isotropy group of μ, and $\mathcal{O} \simeq G/G_\mu$. The dual projections are

$$h^*: \mathfrak{m}_\mu^* \to \mathfrak{g}^* \quad ; v^*: \mathfrak{g}_\mu^* \to \mathfrak{g}^*$$

and satisfy

$\mathrm{im}(h^*) = \mathfrak{g}_\mu^\circ = T_\mu \mathcal{O}$, the annihilator of $\mathfrak{g}_\mu$;

$\mathrm{im}(v^*) = \mathfrak{m}_\mu^\circ = N_\mu$.

The horizontal lift, $h_e: T_\mu \mathcal{O} \to T_e G = \mathfrak{g}$ is given by

$$h_e(\gamma_{\mathfrak{g}^*}(\mu)) = \gamma \qquad [1.3.4]$$

where the argument of h_e is the infinitesimal generator of the (left) co-adjoint action on $\mathcal{O}$, namely

$$\gamma_{\mathfrak{g}^*}(\mu) = -\mathrm{ad}_\gamma^* \mu \; .$$

The normal bundle N is isomorphic as a vector bundle over $\mathcal{O}$ to the co-adjoint bundle $Ad^*G = (G \times \mathfrak{g}_\mu^*)/G_\mu$ which is associated to the principal G_μ bundle $G \to G/G_\mu \simeq \mathcal{O}$. The isomorphism

$$\lambda: Ad^*G \to N$$

is given by

$$\lambda([g,\alpha]) = (\mathrm{Ad}_{g^{-1}}^{*}\mu \,,\, \mathrm{Ad}_{g^{-1}}^{*} v^{*}\alpha) \in N_{g\mu} = \{g\mu\} \times \mathfrak{g}_{g\mu}^{*}$$

The prescription of §1.2 tells us to put the Poisson structure on Ad*G which is obtained by reducing $G \times \mathfrak{g}_\mu^*$ by the 'diagonal' right G_μ action where the symplectic structure used on $G \times \mathfrak{g}_\mu^*$ is

$$\Omega = \pi^* \omega_{\Theta} - d\langle \alpha, A \rangle.$$

Here ω_Θ is the (-) orbit symplectic form on Θ, and the $\mathfrak{g}_\mu$-valued connection one-form A on G is the one described above. By the connection-induced bracket on N, we then mean the push-forward of the Poisson structure on Ad*G to N by λ.

<u>**Theorem 1**</u>

Molino's Poisson structure on N equals the connection-induced Poisson structure on N.

The symplectic form ω_e which determines the horizontal part of this Poisson bracket (see formula [PB.2*] of §1.2) is given by

$$\omega_e(\mu)(\xi_{\mathfrak{g}^*}(\mu), \gamma_{\mathfrak{g}^*}(\mu)) = \langle \mu + e, [\xi, \gamma] \rangle \qquad \text{[NB1].}$$

The first part of this theorem can be restated as: "$\lambda: \mathrm{Ad}^*G \to N$ is a Poisson isomorphism where N has Molino's Poisson structure." The theorem is proved by using the following two lemmas.

<u>LEMMA 1.</u> The map $i(g,\alpha) = TL_{g^{-1}}^*(\mu + v^*\alpha)$ is a symplectic embedding of $G \times \mathfrak{g}_\mu^*$, with symplectic form Ω, into T*G with its canonical symplectic form. This embedding satisfies

$$\mathrm{im}\,T_{(g,\alpha)} i \subset [T_\beta(\beta \cdot G)]^{\square} \qquad \text{[L1]}$$

Here $\beta = i(g,\alpha) \in T_g^*G$, the G orbit $\beta \cdot G$ is with respect to the right action of G: $\beta \cdot h = TR_{h^{-1}}^* \beta$, and "$\square$" denotes the symplectic orthogonal complement in T*G. This embedding is G_μ-equivariant, where the G_μ action on T*G is the restriction of the right G action, and the G_μ action on $G \times \mathfrak{g}_\mu^*$ is the 'diagonal' action.

The abstract setting for the second lemma is as follows. (S,Ω)

is a Hamiltonian K space, (P,ω) a Hamiltonian G space, K is a closed subgroup of G and $i: S \longrightarrow P$ is a K-equivariant symplectic embedding. S/K and P/G are assumed to be manifolds. The respective Poisson projections are denoted $\pi: S \to S/K$ and $\Pi: P \to P/G$.

LEMMA 2.

Under the above hypothesis, if $[\mathrm{Im}\, T_s i]^{\square} \subset T_p G\cdot p$, whenever $p = i(s)$, then the induced map $[i]: S/K \to P/G$ is a Poisson map.

Before proving these lemmas, we use them to provide the

Proof of Theorem 1.

Set $S = G \times \mathfrak{g}_\mu{}^*$, $P = T^*G$. Lemma 1 states that i satisfies the hypothesis of Lemma 2. Hence

$$[i]: \mathrm{Ad}^*G \to \mathfrak{g}_+{}^*$$

(the subscript "–" denotes the minus Lie-Poisson structure) is a local Poisson isomorphism. Chasing the definitions, one sees that

$$[i]\circ\lambda^{-1}: N \to \mathrm{Ad}^*G \to \mathfrak{g}^*$$

is equal to the exponential map which Molino uses. It follows that λ is a (local) Poisson isomorphism. This proves the first part of

Theorem 1.

To prove the formula [NB1], note from [1.3.3] and [1.3.4] that $\langle e,F(\mu)\rangle(\xi_{\mathfrak{g}^*},\gamma_{\mathfrak{g}^*}) = -\langle e,[\xi,\gamma]\rangle$. The formula now follows directly from the facts that the + orbit symplectic form is $\omega(\mu)(\xi_{\mathfrak{g}^*},\gamma_{\mathfrak{g}^*})$ and the definition, $\omega_e(\mu) = \omega(\mu) - \langle e,F\rangle(\mu)$, of ω_e. ●

Proof of Lemma 1.

As noted in the previous section, a two-form of the type Ω is automatically nondegenerate, at least in a neighborhood of $G\times\{0\}$. So, if we show that $i^*\omega = \Omega$, we will have shown that the embedding i is symplectic, at least in this neighborhood. Here ω denotes the canonical two-form on T^*G.

It is convenient to think of i as the composition

$$G\times\mathfrak{g}_\mu{}^* \xrightarrow{\ j\ } G\times\mathfrak{g}^* \xrightarrow{\ L\ } T^*G$$

namely

$$(g,\alpha) \longmapsto (g,\mu+v^*\alpha) \longmapsto TL_{g^{-1}}{}^*(\mu+v^*\alpha).$$

The pull-back of the canonical one-form θ on T^*G by the left trivialization map L, is

$L^*\Theta(g,\beta) = \langle \beta,\varphi(g)\rangle$

where φ is the Maurer-Cartan form on G described earlier (pulled back to $G\times\mathfrak{g}^*$ by the projection onto the first factor). So

$i^*\Theta(g,\alpha) = \langle v^*\alpha,\varphi(g)\rangle + \langle \mu,\varphi(g)\rangle$

$= \langle \mu,\varphi(g)\rangle + \langle \alpha, v\cdot\varphi(g)\rangle$

$= \langle \mu,\varphi(g)\rangle + \langle \alpha, A(g)\rangle.$

Hence,

$i^*\omega(g,\alpha) = -i^*d\Theta(g,\alpha)$

$= -d\langle \mu,\varphi(g)\rangle - d\langle \alpha, A(g)\rangle$

To prove our first claim, it now suffices to show that $\pi^*\omega_{\mathcal{O}} = -d\langle \mu,\varphi(g)\rangle$. Let $\hat{i}$ denote the restriction of i to Gx{0}. Clearly $\hat{i}^*\omega = -d\langle \mu,\varphi(g)\rangle$. The image of $\hat{i}$ is the left-invariant form generated by μ, which also equals $J_r^{-1}(\mu)$ where $J_r(\beta_g) = TL_g{}^*\beta_g$ is the momentum map for the right action of G. So we must show that $\hat{i}^*\omega = \pi^*\omega_{\mathcal{O}}$. But this is the defining equation for the reduced symplectic structure on $\mathcal{O} \simeq J_r^{-1}(\mu)/G_\mu$, which is known to be the symplectic structure $\omega_{\mathcal{O}}$.

Our second claim, inclusion [L1] follows immediately from the canonical dual pair structure

$$\mathfrak{g}_-^* \xleftarrow{\ J_r\ } T^*G \xrightarrow{\ J_l=\pi\ } \mathfrak{g}_+^*$$

on T*G. Here J_l is **right** trivialization,

$J_l(\beta_g) = TR_g{}^*\beta_g,$

which is the projection for the right G action (and the momentum map for the **left** G action). If we freeze $\alpha \in \mathfrak{g}_\mu^*$, we have $i(G\times\{\alpha\}) = J_r^{-1}(\mu+v^*\alpha)$. Thus, $\operatorname{im} Ti_{(g,\alpha)} \supset T_{i(g,\alpha)} J_r^{-1}(\mu+v^*\alpha) = \ker T_{i(g,\alpha)} J_r$. Taking the symplectic orthogonal complements of both sides of this inclusion yields the desired result:

$[\operatorname{im} Ti_{(g,\alpha)}]^{\square} \subset [\ker T_{i(g,\alpha)} J_r]^{\square} = \ker T_{i(g,\alpha)}\pi$

where the final equality is true by definition of dual pair.

The final claim of this lemma, that i is G_μ equivariant, is a straightforward verification.●

<u>Proof of Lemma 2.</u>

The relevant commutative diagram is

$$\begin{array}{ccc} S & \xrightarrow{\;\; i \;\;} & P \\ \Big\downarrow \pi & & \Big\downarrow \Pi \\ S/K & \xrightarrow{\;\; [i] \;\;} & P/G \end{array}$$

diagram 1.3.1

Let f and g be functions on P/G. We must show that

$$\{f\circ[i], g\circ[i]\}_{S/K} = \{f, g\}_{P/K}\circ[i] \;.$$

This is equivalent, by definition of reduced Poisson structures, to showing that

$$\Omega(X_{f\circ[i]\circ\pi}, X_{g\circ[i]\circ\pi}) = \omega\,(X_{f\circ\Pi}, X_{g\circ\Pi}).$$

But $\Omega = i^*\omega$, so

$$\Omega(X_{f\circ[i]\circ\pi}, X_{g\circ[i]\circ\pi}) = \omega(Ti X_{f\circ[i]\circ\pi}, Ti X_{g\circ[i]\circ\pi})$$

Hence it suffices to show that

$$X_{f\circ\Pi} = Ti\cdot X_{f\circ[i]\circ\pi} \qquad [1.3.2]$$

First, we show that $X_{f\circ\Pi}$ is in imTi. Clearly $X_{f\circ\Pi} \in [\ker T\Pi]\square$, since if $v \in \ker T\Pi$, then $\omega(X_{f\circ\Pi}, v) = df\cdot T\Pi\cdot v = 0$. But $\ker T\Pi = T_pGp \supset [\mathrm{im}Ti]\square$, so $X_{f\circ\Pi} \in [\ker T\Pi]\square \subset \mathrm{im}Ti$. If we can now show that

$$\omega(X_{f\circ\Pi}, Tiv) = \omega(Ti\cdot X_{f\circ[i]\circ\pi}, Tiv), \quad \text{for all } v \in T_sS \qquad [1.3.3]$$

then we will have shown [1.3.2] (since imTi is a symplectic subspace of T_pP, so that the pairing defined by $\omega|_{\mathrm{im}Ti}$ is nondegenerate). But

$$\begin{aligned} \omega(TiX_{f\circ[i]\circ\pi}, Tiv) &= i^*\omega(X_{f\circ[i]\circ\pi}, v) \\ &= \Omega(X_{f\circ[i]\circ\pi}, v) \\ &= d(f\circ[i]\circ\pi)\cdot v \\ &= d(f\circ\Pi\circ i)\cdot v \\ &= d(f\circ\Pi)\cdot Ti\cdot v \\ &= \omega(X_{f\circ\Pi}, Ti\cdot v). \end{aligned}$$

This completes the proof of lemma and hence of Theorem 1. ●

Remarks.

1. Up to coverings, the investigation just completed is the investigation of Poisson vector bundles whose base space is a symplectic homogeneous space G/K with associated principal bundle $G \to G/K$ admitting a G-invariant connection. This is because all homogeneous symplectic spaces are coverings of co-adjoint orbits $\mathcal{O} = G/G_\mu$ (see Guillemin and Sternberg [1984] for a proof of this).

2. Since the exponential map $N \to \mathfrak{g}_+^*$ is a Poisson map, abstract theory tells us that it is the momentum map for a left action on N. This action on N is the one which is induced by the co-adjoint action on $\mathfrak{g}^*$.

Conversely, we could have noted that this G action is Hamiltonian with respect to the connection-induced structure on N. Guillemin and Sternberg [1985, ch. 40] did this and calculated the corresponding momentum map. (However, they did not state that $Ad^*(G) = N(\mathcal{O})$.) By inspection, their expression for the momentum map, is Molino's expression for the exponential map. As a momentum map then, it is automatically a Poisson map onto $\mathfrak{g}_+^*$. This provides an alternative proof of Theorem 1.

3. Consider the case where G is compact. Then $\mathfrak{g}$ has a bi-invariant positive definite inner product which induces an isomorphism $\mathfrak{g}^* \to \mathfrak{g}$ intertwining the co-adjoint and the adjoint actions. So an adjoint orbit $\mathcal{O}$ has both a symplectic structure and a Riemannian structure as a submanifold of $\mathfrak{g}$. The curvature F (of

formula [NB1]) is expressible in terms of the Riemannian curvature of $\mathcal{O}$ (if for no other reason than that both connections are G-invariant). The Riemannian volume form is a constant multiple of the Liouville form $\omega_{\mathcal{O}}^k$, $k = \dim\mathcal{O}$, since both forms are G-invariant. (For principal orbits the constant of proportionality is $1/n!(\theta^1(\mu)\theta^2(\mu)\ldots\theta^k(\mu))$ where $\mu \in \mathcal{O}$ and the θ^i are the roots which are elements of $\mathfrak{g}_\mu^*$.) In case $\mathcal{O}$ is a principal orbit, then $N(\mathcal{O}) \approx \mathcal{O} \times \mathfrak{t}$, where $\mathfrak{t} = \mathfrak{g}_\mu$ is the Lie algebra of the maximal torus G_μ. And for $e \in \mathfrak{t}$ small enough, $\exp^* dvol(\mu, e)$ is a constant multiple of $(\omega_{\mathcal{O}} - \langle e, F\rangle)^k d^{n-k}t$ where dvol is the volume form on $\mathfrak{g}$ and $d^{n-k}t$ the volume form on $\mathfrak{g}_\mu$ ($n = \dim\mathfrak{g}$, $n-k = \mathrm{rank}\,\mathfrak{g}$). If our curvature conjecture is correct, then this seems to say that the pull-back of the volume form is expressible completely in terms of the metric and the Riemannian curvature of $\mathcal{O}$. Integrating this over $\mathcal{O} \times \{e : \|e\| < a\}$, we should obtain a special case of the Weyl Tube Theorem [Weyl, 1939], which says that the volume of this tube is polynomial of degree n-k in a, whose coefficients are integrals of traces of the Riemannian curvature of $\mathcal{O}$.

4. The symplectic embedding $i: G \times \mathfrak{g}_\mu^* \hookrightarrow T^*G$ is closely related to the embedding $j: B \times \mathfrak{g}^* \to B^\# \times \mathfrak{g}^* \to T^*B$ of remark 5, §1.2, where in our case $B = G$, $\mathfrak{g} = \mathfrak{g}_\mu$, and (under left trivialization) $B^\# = G^\# \simeq G \times \mathfrak{m}_\mu^*$. To see this relation, note that i is the composition

$$G \times \mathfrak{g}_\mu^* \xrightarrow{\text{id.} \times 0 \times \text{id.}} G \times \mathfrak{m}_\mu^* \times \mathfrak{g}_\mu^* \simeq G^\# \times \mathfrak{g}_\mu^* \xrightarrow{\simeq} G \times \mathfrak{g}^* \xrightarrow[\simeq]{\text{shift by } \mu} G \times \mathfrak{g}^* \simeq T^*G.$$

where "shift by μ" means the map $(g,\beta)\mapsto(g,\mu+\beta)$. And note that j is the same composition, except with the "shift by μ" map omitted. Thus j = (shift by $-\mu$)'i. The push-forward of the canonical two-form ω_G on $T^*G \cong G\times\mathfrak{g}^*$ by the shift by $-\mu$ map is $\omega_G+\pi^*\omega_\Theta$, which corresponds to the general $\omega_B+(pr)^*\omega_S$ of remark 5, §1.2. This demonstrates that j is a symplectic embedding (with degeneracies) of $G\times\mathfrak{g}_\mu^*$ into T^*G with the shifted symplectic form $\omega_G+\pi^*\omega_\Theta$.

<u>Singularities of the Poisson Structure.</u>

The Poisson structure on N blows up precisely at those points (μ,e) where $\omega_e(\mu)$ of formula [NB1] above is degenerate. This was noted in remark 1 of the previous section. The blow-up is as $1/\epsilon$ as $(\mu,(1+\epsilon)e)$ approaches (μ,e), since $\omega_{(\mu,(1+\epsilon)e)}$ degenerates as ϵ. Since Molino's Poisson structure equals the connection-induced structure, this leads us to formulate the

Singularity Theorem .

The critical points of $\exp:N(\Theta)\longrightarrow\mathfrak{g}^*$, Θ a reductive orbit are precisely the points where $\omega_e(\mu)$ is degenerate , and hence where the Poisson structure blows up. As $(\mu,(1+\epsilon)e)$ approaches a critical point (μ,e) the Poisson tensor blows up as $1/\epsilon$.

We will prove the theorem after presenting some examples. As an illustration that this theorem is not obvious, consider the following example where the exponential map has critical points but the induced Poisson structure on the normal bundle easily extends through these critical points. Let P be the Poisson manifold S^2 with the trivial, i.e. identically zero, Poisson structure. Let L be the symplectic leaf {n} where n is the north pole. The exponential map for the standard metric on S^2 simultaneously linearizes the leaf {n}, since the single fibre of its normal bundle $N = T_nS^2$ and S^2 both have the trivial Poisson structure. Thus the Poisson structure induced on N by the exponential map is trivial. Even though the map has a critical value at the south pole s, the induced Poisson structure on N can be continued trivially across the circle of points $\exp^{-1}(s)$ in order to make a smooth globally defined Poisson structure. One can imagine concocting similar, less trivial examples by relating, at the critical values of exp, the singular directions of the exponential map to sub-Casimirs on the leaf.

Before going ahead to the proof of the theorem, it is instructive to understand the following example.

The example $N(S^2)$.

The standard metric on $\mathbb{R}^3 \cong \mathfrak{so}(3)^*$ is bi-invariant and we use it to identify the abstract normal bundle with the geometric normal bundle of S^2. The co-adjoint orbit through $\mu \in \mathbb{R}^3$ is the sphere $S^2(\|\mu\|)$ containing μ. Take $\|\mu\| = 1$ so that we have the standard sphere. As a homogeneous space $S^2 = SO(3)/SO(2)$and we may think of the SO(3) as the orthonormal frame bundle for S^2. One checks that the invariant connection constructed above for the general case $G \rightarrow \Theta$ is in our case the Riemannian connection for $SO(3) \rightarrow S^2$. The curvature of this connection is $F = -\omega$, where ω is the standard symplectic form on S^2. Since $\ker\omega = 0$, the kernel of the curvature is <u>not</u> co-isotropic, so by Theorem 2 of the preceding section there will be cross-curvature terms in the formula [PB.2] for the brackets on $N = N(S^2)$. Now N is the trivial bundle, $S^2 \times \mathbb{R}$ where the fibre $\mathbb{R}$ has the trivial Poisson structure. So according to Theorem 1 of the preceding section the Poisson tensor on N at (μ,e), which is a linear map $T^*_\mu S^2 \times \mathbb{R} \rightarrow T_\mu S^2 \times \mathbb{R}$, is given by $J_e \oplus 0$ where J_e is the inverse to $\omega_e = \omega - eF = (1+e)\omega$. Thus the Poisson structure blows up as $1/\epsilon$ as $e = -1+\epsilon$ approaches -1. [This calculation can easily be done by hand by using the exponential map $N \rightarrow \mathbb{R}^3$ to pull back the Lie-Poisson structure on $\mathbb{R}^3$.] Note that the points $(\mu,-1)$ are the critical points of the exponential map. All these points are mapped to the origin under the exponential map. The same situation, rescaled, occurs for a sphere of radius r: the Poisson structure is singular at the focus points $(\mu,-r)$ with singularity of the same form.

Note that by putting the product Poisson structure on $N = S^2 \times \mathbb{R}$ we would have obtained another realization of N, but it would not have been Poisson isomorphic to the one just described.

Proof of the Singularity Theorem.

We will prove the theorem through a series of observations.

<u>Observation 1.</u> **If** $e \in N_\mu = \mathfrak{m}_\mu^*$ **and** $\gamma \in \mathfrak{m}_\mu$ **then** $ad_\gamma^*(\mu+e) \in \mathfrak{g}_\mu^*$.

For if $\psi \in \mathfrak{g}_\mu$, then

$$\begin{aligned} ad_\gamma^*(\mu+e)\cdot\psi &= \langle \mu, [\gamma,\psi]\rangle + \langle e, [\gamma,\psi]\rangle \\ &= \langle e, [\gamma,\psi]\rangle \qquad (\text{since } \psi \in \mathfrak{g}_\mu) \\ &= 0 \quad (\text{ since } [\gamma,\psi] \in \mathfrak{m}_\mu \text{ and } e \in \mathfrak{m}_\mu^*). \end{aligned}$$

<u>Observation 2.</u> $T_{(\mu,e)}\exp$ **is singular if and only if there is a non-zero** $\gamma \in \mathfrak{m}_\mu$ **such that** $ad_\gamma^*(\mu+e)$ **is zero**.

To see this, we decompose the differential of exp. Note that $\gamma \mapsto \gamma_{\mathfrak{g}^*} = ad_\gamma^*\mu$ identifies $\mathfrak{m}_\mu$ with $\mathfrak{g}_\mu^* = T_\mu\Theta$. (This is because $\mathfrak{m}_\mu$ is the complementary subspace to $\mathfrak{g}_\mu$ which is the kernel of this map.) Using the connection we can decompose $T_{(\mu,e)}N$ as $\mathfrak{m}_\mu \oplus \mathfrak{m}_\mu^*$ where the first factor represents horizontal vectors and the second factor represents vertical vectors. From the formula for exp, we see under this identification that $T\exp_{(\mu,e)}\cdot 0 \oplus \alpha = \alpha$. To calculate the horizontal derivative, note that the horizontal lift of the curve $Ad_{\exp t\gamma}^*\mu$, $\gamma \in \mathfrak{m}_\mu$, is the curve $(Ad_{\exp t\gamma}^*\mu, Ad_{\exp t\gamma}^*e)$. Therefore, $T\exp_{(\mu,e)}\cdot\gamma \oplus 0 = d/dt|_{t=0}(Ad_{\exp t\gamma}^*\mu + Ad_{\exp t\gamma}^*e) = ad_\gamma^*(\mu+e)$. From

<u>observation 1</u>, we know that $ad_\gamma^*(\mu+e) \in \mathfrak{g}_\mu^*$. So as a linear transformation $\mathfrak{m}_\mu \oplus \mathfrak{m}_\mu^* \to \mathfrak{g}_\mu^* \oplus \mathfrak{m}_\mu^*$, $Texp_{(\mu,e)}$ has block matrix form

$$\begin{matrix} ad_{(\cdot)}^*(\mu+e) & 0 \\ 0 & I \end{matrix}$$

and hence is inverible if and only if $\gamma \longmapsto ad_\gamma^*(\mu+e)$ is invertible.

<u>Observation 3.</u> $\omega_e(\mu)$ **is degenerate if and only if there is a non-zero** $\gamma \in \mathfrak{m}_\mu$ **such that** $ad_\gamma^*(\mu+e) = 0$.

Again identifying $T_\mu\Theta$ with $\mathfrak{m}_\mu$ formula [NB1] reads $\omega_e(\mu)(\gamma,\xi) = \langle \mu+e, [\gamma,\xi] \rangle$, for $\gamma,\xi \in \mathfrak{m}_\mu$. Thus $\omega_e(\mu)$ is non-degenerate if and only if for all $\gamma \in \mathfrak{m}_\mu$, $ad_\gamma^*(\mu+e)$ is nonzero **when restricted to** $\mathfrak{m}_\mu$. But according to <u>observation 1</u>, the latter is equivalent to $ad_\gamma^*(\mu+e)$ being non-zero.

Comparing <u>observation 2</u> with <u>observation 3</u>, we see that we have proved the theorem. ●

<u>Focal Points and Regular Orbits.</u>

A **focal point** is a critical value of exp. Using the fact that the symplectic leaves for the connection-induced Poisson structure on $N(\Theta)$ all have dimension $\geq \dim\Theta$, using the Singularity Theorem, and using the fact that exp is a Poisson map where it is not critical, we can deduce that

$$\bigcup_{\dim\Theta^\sim \leq \dim\Theta} \Theta^\sim \subset \text{the set of focal points} .$$

In the case where Θ is regular (and reductive) we can obtain the sharper result

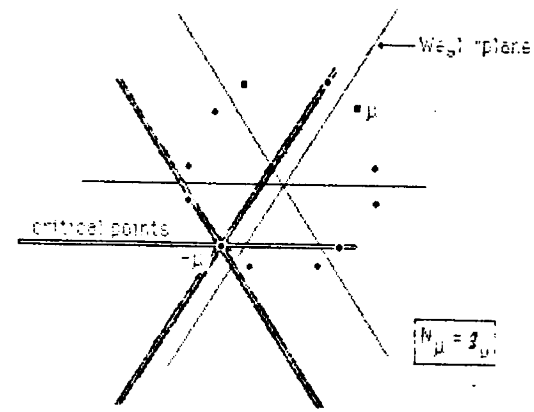
$$\bigcup_{\dim\Theta^\sim < \dim\Theta} \Theta^\sim = \text{the set of focal points} \qquad [1.3.5].$$

To see this, recall that in the regular case that $\mathfrak{g}_\mu$ is Abelian. A calculation shows that this implies that $\mathfrak{g}_\mu \subset \mathfrak{g}_{\mu+e}$, for $e \in \mathfrak{m}_\mu^*$. Thus $\dim\mathfrak{g}_{\mu+e} - \dim\mathfrak{g}_\mu = \dim(\mathfrak{m}_\mu \cap \mathfrak{g}_{\mu+e})$. But $\dim\mathfrak{g}_\alpha = \operatorname{codim}\Theta_\alpha$, for any $\alpha \in \mathfrak{g}^*$. It follows that $\operatorname{codim}\Theta_{\mu+e} < \operatorname{codim}\Theta_\mu$ if and only if $\dim(\mathfrak{m}_\mu \cap \mathfrak{g}_{\mu+e}) > 0$. But observation 3 of the preceding paragraph says that this latter condition is precisely the condition that $T_{(\mu,e)}\exp$ be singular, since $\mathfrak{m}_\mu \cap \mathfrak{g}_{\mu+e} = \{\gamma \in \mathfrak{m}_\mu : ad_\gamma^*(\mu+e) = 0\}$. So we have demonstrated the equality [1.3.5].

In the case where G is compact the non-regular orbits are those which lie on Weyl hyperplanes. As usual in the compact case, we use a bi-invariant positive definite inner product to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. This identification intertwines the adjoint with the co-adjoint action. Under this identification $N_\mu = \mathfrak{g}_\mu$ which is the Lie algebra of the maximal torus G_μ, and $T_\mu\mathcal{O} = \mathfrak{m}_\mu = \mathfrak{g}_\mu^\perp$. It follows from [1.3.5] that

{critical points of exp} = $\{(\mu,e) \in N : \mu+e \notin$ a Weyl chamber in $\mathfrak{g}_\mu\}$.

The picture of the critical points in N_μ for μ a regular point in $\mathfrak{su}(3)$ is



§1.4. The Gauge Group.

The gauge group $\mathcal{G}$=Aut(B) is the group of bundle automorphisms of B which cover the identity on the base X. $\mathcal{G}$ acts naturally on both the reduced phase space T*B/G and on the co-adjoint bundle Ad*(B*).

An $\eta \in \mathcal{G}$ acts on T*B/G according to

$$\eta\cdot[\alpha_b] = [T^*\eta_b{}^{-1}(\alpha_b)]$$

(This transformation is well defined because η commutes with the right action of G, so that the equivalence class on the right hand side of this equation is independent of the representative, α_b, picked from the left hand side.) We will denote this transformation by $[T^*\eta^{-1}]$. It is a canonical transformation since $T^*\eta^{-1}$ is a canonical transformation of T*B.

The automorphism η_{Ad^*} which η induces on Ad*(B) is described as follows. η induces a bundle automorphism on any pull-back bundle of B, in particular it induces the automorphism $\eta^\#$ of $B^\# \subset T^*B$, the pull-back of B to T*X. One easily checks that $\eta^\#$ is the restriction of $T^*\eta^{-1}$ to $B^\#$. In turn, $\eta^\#$ induces a vector bundle automorphism on any vector bundle associated to $B^\#$. In particular it induces the vector bundle automorphism η_{Ad^*} on $Ad^*(B^\#) = B^\# \times_G \mathfrak{g}^*$. By construction,

$$\eta_{Ad^*} = [\eta^\# \times id] .$$

The notation means

$$\eta_{Ad^*}[\alpha,Q] = [\eta^\#(\alpha),Q]$$
$$= [T^*\eta^{-1}(\alpha),Q]$$

where $[\alpha,Q] \in B^\# \times_G \mathfrak{g}^*$ denotes the equivalence class containing $(\alpha,Q) \in B^\# \times \mathfrak{g}^*$.

On the unreduced level we have the diagram

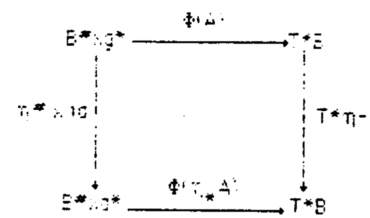


diagram 1.4.1

This diagram commutes:

$$T^*\eta^{-1}\Phi(A)(B_b,\mu) = T^*\eta^{-1}\cdot(B_b + A_b{}^*\mu)$$
$$= T^*\eta^{-1}\cdot B_b + T^*\eta^{-1}A_b{}^*\mu$$
$$= T^*\eta^{-1}\cdot B_b + (\eta_* A_{\eta(b)})^*\mu$$

$$= \Phi(\eta_* A)(\eta^*(B_b), \mu).$$

Put the minimally coupled symplectic forms corresponding to A and $\eta_* A$ on $B^* \times \mathfrak{g}^*$. Then, by definition of these forms, the above diagram is a commutative diagram of symplectic manifolds. So if we divide by G we get the following commutative diagram of Poisson manifolds:

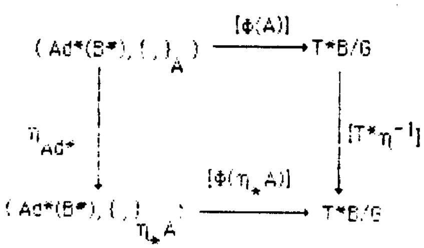


diagram 1.4.2

Remark. In the more general setting of §1.2 an analog of T^*B may not exist but η_{Ad^*} is still a Poisson map. To see this, recall the general setting. We have a principal G bundle $P \to S$ over a symplectic manifold (S, ω_S) with a connection Γ. The Poisson bracket $\{\ ,\ \}_\Gamma$ on $Ad^*(P) = P \times_G \mathfrak{g}^*$ is obtained by reducing the symplectic form

$\omega_\Gamma = \pi^* \omega_S - d\langle \mu, \Gamma \rangle$ on $P \times \mathfrak{g}^*$ by the 'diagonal' action of the structure group G. An automorphism η of P induces the automorphism $\eta_{Ad^*} = [\eta \times id.]$ of $Ad^*(P)$. [When P was the pull-back bundle $B^* \to T^*M = S$ of a bundle $B \to M$ as in the beginning of this section, we took Γ to be A^* the pull-back of a connection A on B, and our present η was called η^*.] Thus to show that η_{Ad^*} is a Poisson map between $(Ad^*(P), \{\ ,\ \}_\Gamma)$ and $(Ad^*(P), \{\ ,\ \}_{\eta\Gamma})$, it suffices to show that ω_Γ is the pullback of $\omega_{\eta\Gamma}$ by $\eta \times id.$:

$$\begin{aligned}
(\eta \times id.)^* \omega_{\eta\Gamma} &= \eta^* \pi^* \omega_{T^*M} - (\eta^* \times id.)^* d\langle \mu, \eta_* \Gamma \rangle \\
&= (\pi^* \eta)^* \omega_{T^*M} - d(\eta^* \times id.)^* \langle \mu, \eta_* \Gamma \rangle \\
&= \pi^* \omega_{T^*M} - d\langle id.^* \mu, \eta^* \eta_* \Gamma \rangle \\
&= \omega_\Gamma. \quad \bullet
\end{aligned}$$

The Momentum Map for the $\mathcal{G}$ action.

We now calculate the momentum map j for the action of the gauge group on the reduced cotangent bundle. (There is no point in trying to calculate a momentum map for the $\mathcal{G}$ action on $Ad^*(B^*)$ since, by diagram 1.4.2, this action is not canonical relative to **any** fixed bracket on $Ad^*(B^*)$ which corresponds to a fixed connection on B.)

The Lie algebra of $\mathcal{G}$ is $\Gamma(Ad(B))$, the space of sections of the adjoint bundle $B \times_G \mathfrak{g}$. This Lie algebra will be denoted $\mathcal{G}$. It is naturally isomorphic to the space of equivariant functions $B \to \mathfrak{g}$ (where the action on $\mathfrak{g}$ is the adjoint action, made into a right action). The isomorphism takes the equivariant function $\xi: B \to \mathfrak{g}$ to the section

$$x = \pi(b) \mapsto [b, \xi(b)] = \xi(x) \in \mathcal{G}.$$

Under this identification, the infinitesimal action on B corresponding to ξ is given by

$$\xi_B(b) = \sigma_b \xi(b) \in T_b B. \qquad [1.4.1]$$

(Recall that $\sigma_b: \mathfrak{g} \to T_b B$ is the infinitesimal generator map for the right G action.)

The momentum map j must be a function with values in $\mathcal{G}^* = \Gamma^*(Ad(B)) \simeq \Gamma(Ad^*(B)) \bullet C^\infty(X)^*$. Here $C^\infty(X)^*$ is the space of distributions on X. The pairing between an $\xi \in \Gamma(Ad(B))$ and $\varphi \bullet \lambda \in \Gamma(Ad^*(B)) \bullet C^\infty(X)^*$ is

$$\langle\langle \varphi \bullet \lambda, \xi \rangle\rangle = \langle \lambda, \varphi \cdot \xi \rangle.$$

Here $\varphi \cdot \xi \in C^\infty(X)$ denotes the function obtained using the pointwise pairing between Ad*(B) and Ad(B) and the outer brackets, "< , >" on the right hand side denote the pairing between $C^\infty(X)^*$ and $C^\infty(X)$ and.

The formula for the momentum map $j^{\sim}$ for the action of $\mathcal{G}$ on T*B is

$$\langle\langle j^{\sim}(\alpha_b), \xi \rangle\rangle = \langle \alpha_b, \xi_B(b) \rangle \quad \text{(pairing between T*B and TB).}$$

From [1.4.1] we see that

$$\langle\langle j^{\sim}(\alpha_b), \xi \rangle\rangle = \langle \sigma_b{}^* \alpha_b, \xi(b) \rangle \quad \text{(pairing between } \mathfrak{g}^* \text{ and } \mathfrak{g}).$$

$$= [b, j(\alpha_b)] \cdot [b, \xi(b)] \text{ (pairing between Ad*B and AdB)}$$

$$= \langle\langle [b, \sigma_b{}^*(\alpha_b)] \bullet \delta(x - x'), \xi \rangle\rangle$$

$$\text{(pairing between } \Gamma(Ad(B)) \text{ and } \Gamma(Ad^*(B)) \bullet C^\infty(X)^*).$$

so that

$$j^{\sim}(\alpha_b) = [b, \sigma_b{}^*(\alpha_b)] \bullet \delta(\pi(b) - x').$$

Note that this function is invariant under the action of G. The δ function factor is clearly invariant, since π is G-invariant. The first factor is also G-invariant: $[bg, \sigma_b{}^*(T^*R_{g^{-1}}\alpha_b)] = [bg, Ad_g{}^* \sigma_b{}^* \alpha_b] = [b, \sigma_b{}^* \alpha_b]$. (Recall that σ^* is the equivariant momentum map for the right G action on T*B.) So $j^{\sim}$ descends to the quotient, T*B/G. Let j denote the function on the quotient. It is automatically the momentum map for the action of G on the reduced cotangent bundle. Summarizing

PROPOSITION. The momentum map for the $\mathcal{G}$-action on T^*B/G is given by

$$J(\alpha_b) = [b, \sigma_b{}^*\alpha_b] \bullet \delta(x-x') \ .$$

Local and physical expressions for actions and the momentum map.

From a physical point of view, the gauge group acts on $Ad^*(B^\#)$ by "rotating" the color charge, while leaving the position and velocity fixed. Specifically, an $\eta \in \mathcal{G}$ acts on $Ad^*(B^\#)$ according to the local expression

$$(x, p, Q) \overset{\eta_{Ad^*}}{\longmapsto} (x, p, \tilde{Q}) = (x, p, Ad_{g(x)^{-1}}{}^* Q)$$

where $\eta(x,h) = (x, g(x)h)$ is the local expression for η.

To describe the action of $\mathcal{G}$ on T^*B/G from a physical point of view, recall that connections transform under gauge transformations

according to the local formula $A \mapsto \tilde{A} = gAg^{-1} + gdg^{-1}$. Velocities and rest mass are unchanged by gauge transformations and so the kinetic momenta p are unchanged. Color charges are "rotated" by gauge transformations. Thus canonical momenta must transform according to

$$\begin{aligned} p^{can.} = p + Q\cdot A \longmapsto \tilde{p}^{can.} &= p + \tilde{Q}\cdot\tilde{A} \\ &= p + Ad_{g^{-1}}{}^* Q\cdot(Ad_g A + gdg^{-1}) \\ &= p + Q\cdot A + Q\cdot Ad_{g^{-1}} gdg^{-1} \\ &= p^{can.} + Q\cdot dg^{-1} g \\ &= p^{can.} - Q\cdot g^{-1} dg \ . \end{aligned}$$

The transformation $(x, p^{can.}, Q) \mapsto (x, \tilde{p}^{can.}, \tilde{Q})$ is the coordinate expression for $[T^*\eta^{-1}]$. To see this, note that the transformation was defined so as to make the following square commutative

$$\begin{array}{ccc} & [\Phi(A)] & \\ (x,p,Q) & \longrightarrow & (x, p^{can.}, Q) \\ \tilde{\eta}_{Ad^*} \downarrow & & \downarrow \tilde{\eta} \\ (x,p,\tilde{Q}) & \longrightarrow & (x, \tilde{p}^{can.}, \tilde{Q}) \\ & [\Phi(\eta_* A)] & \end{array}$$

All the maps in this diagram, except for possibly the transformation

$\tilde{\eta}$ in question, are the local versions of the corresponding maps in the intrinsic diagram 1.4.2 above. Since both squares commute, this proves that $\tilde{\eta} :(x, p^{can.},Q) \mapsto (x, \tilde{p}^{can.},\tilde{Q})$ is the coordinate version of $[T^*\eta^{-1}]$. (One can also check by direct calculation that this is correct.)

We end this section by stating the local version of our proposition

PROPOSITION. The momentum map for the $\mathcal{G}$-action on T*B/G has local expression:

$$j(x, p^{can.},Q) = Q\delta(x-x') \quad . \qquad [M1]$$

This follows directly from our intrinsic description for j, because the expression for the fibre coordinate of $[b,\sigma_b{}^*\alpha_b] \in Ad^*(B)$ in our local trivialization is Q. (See §1 of this chapter for how a trivialization of B induces trivializations of all the other bundles.) Alternatively, one could check by direct calculation that this local expression satisfies the defining property of a momentum map. This is done using the expression for the brackets in $(x, p^{can.},Q)$, the local expression for the action of the gauge group, and noting from the local expression for j that

$$j^{\xi}((x, p^{can.},Q)) = \langle Q, \xi(x)\rangle \ .$$

From a gauge theorist's point of view this local expression for the momentum map is obvious, since it is well known that the conserved quantity corresponding to the action of the gauge group is the color charge.

§2.1. Wong's Equations.

Some history.

Wong's equations are the equations of motion for a classical colored spinless particle in an external Yang-Mills field A. Wong [1970] derived them by taking classical limits of the quantum mechanical Yang-Mills equations. The equations reduce to the Lorentz equations in the Abelian case. Their physical relevance to the non-Abelian case is debatable, but they seem to be of at least qualitative use. See Arodz [1982] for further discussion, and the generalization to the case where the particles have spin. I recomend Balachandran et al. [1983] as a well-done, detailed treatise on the subject.

Besides Wong's equations, there are various geometric formulations of the motion of such a particle. These are: the formulation of Kaluza-Klein as generalized by Kerner [1968], the formulation of Sternberg [1977], and the formulation of Weinstein [1978]. Weinstein [1978] showed how his formalism is equivalent to Sternberg's. A nice presentation and application of the Weinstein and Sternberg's works is given by Sniatycki [1979].

In Montgomery [1984], I showed how these four formalisms were all equivalent. In so doing, a Hamiltonian structure for Wong's equations was derived. The present section is mostly a revision of this earlier paper. One of the main additions is that more attention is paid to the difference between the relativistic and non-relativistic versions of Wong's equations.

The Equations.

The relativistic Wong's equations are

$$dx^{\mu}/d\tau = u^{\mu} \qquad \text{[Wa]}$$

$$dp_{\mu}/d\tau = Q_a F^a{}_{\mu\beta} u^{\beta} \qquad \text{[Wb]}$$

$$dQ_a/d\tau = -Q_d c^d{}_{ab} A^b{}_{\mu} u^{\mu} \qquad \text{[Wc]}.$$

where

$$p_{\mu} = m u_{\mu}$$

is the particle's relativistic momentum, m its rest mass, u^{μ} its space-time velocity, and τ its proper time. Throughout, Greek indices μ, β, etc. are space-time indices and Roman indices a, b, etc. are for the Lie algebra. The x^{μ} are the particle's space-time coordinates. The Q_a are the particle's color-charges. The $F^a{}_{\mu\beta}$ are the components of the Yang-Mills field strength, i.e. the curvature of A. The $c^d{}_{ab}$ are the structure constants for the structure group G of the theory. The speed of light, Planck's constant, and the coupling constant have all been set equal to unity.

Equations [Wa] and [Wb] for the particle's world-line are the equations of a particle under the influence of a generalized Lorentz force which is parameterized by the color charges Q_a. Equation [Wc] says that the color charges or internal variables are parallel translated over this world line. The Q_a are to be thought of as fibre-coordinates for the co-adjoint bundle $X \times \mathfrak{g}^*$ over space-time X.

Hamiltonian structure.

Let X denote Minkowski space and $B = X \times G$ denote the principal bundle on which A is a connection. The generalization to arbitrary space-times and principal bundles over them affords no difficulties. See remark 2 below.

The variables (x^μ, p_β, Q_a) or simply $(x,p,Q) \in X \times X^* \times \mathfrak{g}^*$ coordinatize the co-adjoint bundle $Ad^*((X \times G)^*) = X \times X^* \times \mathfrak{g}^*$ over $T^*X = X \times X^*$. As such, their brackets were recorded in §1.1. Take as the Hamiltonian the the "kinetic" Hamiltonian

$$h = p_\mu p^\mu/2m .$$

Straightforward calculations show that Hamilton's equations:

$$dx^\mu/d\tau = \{ x^\mu, h\}$$

$$dp_\mu/d\tau = \{ p_\mu, h\}$$

$$dQ_a/d\tau = \{ Q_a, h\}$$

are the relativistic Wong's equations.

It is important to also understand how these are equations on the reduced cotangent bundle T^*B/G. This understanding will be neccessary for the next section on Yang-Mills plasmas. Recall that the reduced cotangent bundle is $T^*B/G = (T^*X \times T^*G)/G \simeq T^*X \times \mathfrak{g}^*$. The canonical map relating co-adjoint bundle variables (x,p,Q) to reduced cotangent bundle variables $(x,p^{can.},Q)$ is the minimal coupling procedure: $p^{can.} = p+Q\cdot A$, (x and Q remain unchanged). This was shown in §1.1. In that section, we also showed that the Poisson bracket in terms of the $(x, p^{can.},Q)$ was a product Poisson bracket: the $(x, p^{can.})$ are canonically conjugate coordinates on the T^*X factor, and the Q have Lie-Poisson brackets for $\mathfrak{g}^*$. The Wong Hamiltonian, rewritten in terms of $(x, p^{can.},Q)$ is

$$h = (p^{can.} - Q\cdot A)^2/2m.$$

Hamilton's equations are then Wong's equations, rewritten in these variables:

$$dx^\mu/d\tau = u^\mu$$

$$dp^{can.}{}_\mu/d\tau = Q_a \partial/\partial x^\mu (A^a{}_\beta) u^\beta$$

$$dQ_a/d\tau = -Q_d c^d{}_{ab} A^b{}_\mu u^\mu$$

where

$$u^\mu = (p - Q\cdot A)^\mu/m\ .$$

This canonical version of Wong's equations was described by Gibbons, Holm and Kupershmidt [1984]. They actually presented the non-relativistic version of these equations, but the canonical formalisms for both the relativistic and the non-relativistic cases are essentially the same, as we will see below.

Non-relativistic and 3+1ed Wong's equations.

The non-relativistic Wong's equations are obtained by replacing τ by t, and replacing the relativistic kinetic momentum $p = mu$ (respectively canonical momentum $p^{can.}$) by the non-relativistic kinetic momentum $\mathbf{p} = m\mathbf{v}$ (respectively non-relativistic canonical momentum $\mathbf{p}^{can.} = m\mathbf{p}+Q\mathbf{A}$), where $\mathbf{v}$ is the standard velocity. The resulting equations are:

$$d/dt(\mathbf{x}) = \mathbf{v}$$

$$d/dt(\mathbf{p}) = Q\cdot(\mathbf{E}+\mathbf{v}\mathbf{x}\mathbf{B})$$

$$d/dt(Q) = -ad^*(A_0+\mathbf{A}\cdot\mathbf{v})Q\ .$$

These are the equations originally written down by Wong [1970]. Here the spatial Yang-Mills potential $\mathbf{A}$ is defined by $A = A_0dt + \mathbf{A}\cdot d\mathbf{x}$. The "electric" and "magnetic" fields can be defined as in electromagnetism by $F_{\mu\beta}{}^a dx^\mu dx^\beta = \mathbf{E}\cdot d\mathbf{x}dt + (1/2)\epsilon_{ijk}B_i dx^j dx^k$, where i, j, and k are spatial indices.

Note: these equations are **not** equivalent to the relativistic Wong's equations. The relativistic kinetic momentum is $mu = (m\gamma, m\gamma\mathbf{v})$ where $\gamma = dt/d\tau = 1/\sqrt{1-\mathbf{v}^2}$. If we then write out the components of the relativistic Wong's equations we obtain

$$d/dt(\mathbf{x}) = \mathbf{v}$$

$$d/dt(m\gamma\mathbf{v}) = Q\cdot(\mathbf{E}+\mathbf{v}\mathbf{x}\mathbf{B})$$

$$d/dt(m\gamma) = Q\cdot\mathbf{E}\cdot\mathbf{v}$$

$$d/dt(Q) = -ad^*(A_0+\mathbf{A}\cdot\mathbf{v})Q$$

We call these equations the **3+1ed relativistic Wong's equations**.

The Poisson structure for the non-relativistic equations in the $(\mathbf{x},\mathbf{p},Q)$ variables is the same as that for the four-dimensional Wong's equations **provided that A_0 and the electric field are**

zero . That is, the Hamiltonian is $\mathbf{p}^2/2m$ and the variables have the same brackets as above, except that all space-time indices are replaced by purely spatial indices. Note that the **p** brackets are

$$\{p_i, p_j\} = Q \cdot F_{ij} = \epsilon_{ijk} Q \cdot B^k .$$

These are time-independent, **if and only if the magnetic field B is time-independent**. A sufficient condition for this to be the case is our condition that A_0 and the electric field be zero. (This can be seen by using the four-dimensional Bianchi identity, DF = 0.)

The Poisson brackets in $(\mathbf{x}, \mathbf{p}^{can.}, Q)$ variables are always time independent, with the same form as the relativistic version: **x**, $\mathbf{p}^{can}$ canonically conjugate, Q Lie-Poisson and Poisson commuting with **x** and $\mathbf{p}^{can}$. This allows one to keep the canonical formalism for the non-relativistic Wong's equation. These equations, rewritten in canonical variables, are

$$d/dt(\mathbf{x}) = \mathbf{v}$$

$$d/dt(p^{can.}{}_i) = Q_d[(\partial A^d{}_j/\partial x^i)v^j + \partial A^d{}_0/\partial x^i]$$

$$d/dt(Q_a) = -Q_d c^d{}_{ab}(\mathbf{A}^b \cdot \mathbf{v} + A^b{}_0)$$

where $m\mathbf{v} = \mathbf{p}^{can} - Q \cdot \mathbf{A}$

If one takes as Hamiltonian the (generally time-dependent) Hamiltonian $h = (1/2m)(\mathbf{p}^{can.} - Q \cdot \mathbf{A})^2 - Q \cdot A_0$ then Hamilton's equations are the non-relativistic Wong's equations. Because the $(\mathbf{x}, \mathbf{p}^{can.}, Q)$ brackets are time and connection-independent, they seem to be indispensable in studying the canonical structure of Yang-Mills plasmas (next section) where the connection in general varies with time.

Remarks

1. To get the 3+1ed relativistic Wong's equations in the $(\mathbf{x}, \mathbf{p}^{can.}, Q)$ variables as Hamilton's equations, use the same bracket but use the Hamiltonian $(1/m)[(\mathbf{p}^{can} - Q \cdot \mathbf{A})^2 + m^2]^{1/2} + Q \cdot A_0$. The variable $\mathbf{p} = \mathbf{p}^{can} - Q \cdot \mathbf{A}$ is now the spatial component of the relativistic momentum. See Barut [1979] or Bialnyicki-Birula, Hubbard and Turski [1984] for the Abelian case.

2. If the space-time (or space) X is not flat, then the only change to Wong's equations is to the equations [Wb] where the geodesic correction term $-(1/2m)[\partial g^{\alpha\beta}/\partial x^\mu] p_\alpha p_\beta$ must be added to get the correct evolution for the velocity. Here g is the metric on X. The Hamiltonian structure of the equations remains the same: that is, the local expressions for the Poisson bracket relations are the same and the Hamiltonian is $(1/2m) g^{\alpha\beta} p_\alpha p_\beta$. Also, the bundle B need not be trivial. The local expressions for the equations, and their

Hamiltonian structure remains identical.

The Hamiltonians on $Ad^*(B^*)$ or T^*B/G which we used are obtained by pulling back the kinetic Hamiltonian $(1/2m)g^{\alpha\beta}p_\alpha p_\beta$ on T^*X by the appropriate projection (see diagram 3, ch.1, §1). Due to the nature of these projections, the Hamiltonian on the co-adjoint bundle is "universal", i.e. connection independent, whereas the one on the reduced cotangent bundle depends on the choice of connection. This procedure for obtaining the Hamiltonians was suggested by Weinstein [1978]. A different procedure for obtaining Hamiltonians was suggested by Kaluza and Klein as generalized by Kerner [1968]. In the next paragraph we show that these other Hamiltonians also produce Wong's equations.

Relationship to Kaluza-Klein and Kerner.

The Kaluza-Klein formalism , as generalized by Kerner [1968], states that the classical "path" of our colored particle is a geodesic on the principal bundle B. The metric on B is put together from the metric g on X and a bi-invariant metric γ on G by using the connection to declare that horizontal and vertical vectors are perpendicular. The geodesic flow of this metric is generated by the Hamiltonian on T^*B whose local expression is

$$h(x,p,g,Q) = (1/2m)g^{\alpha\beta}(x)p_\alpha p_\beta + (1/2)\gamma^{ab}Q_aQ_b$$

where as usual

$$p_\alpha = p^{can.}{}_\alpha - Q_aA^a{}_\alpha.$$

Here we have trivialized T^*B as in ch. 1, §1, so that $(x,p^{can.},g,Q) \in T^*X \times G \times \mathfrak{g}^* \simeq_{(locally)} T^*B$. This Hamiltonian is G-invariant, so induces Hamiltonians $h(x, p, Q)$ and $h(x, p^{can.}, Q)$, with the same local expressions, on the co-adjoint and reduced cotangent bundles. These Hamiltonians differ from the ones which we used above by the term $(1/2)\gamma^{ab}Q_aQ_b$. Since γ is bi-invariant, this term is a Casimir on $\mathfrak{g}^*$. (By definition a Casimir is a function whose Poisson bracket with all others function is zero.) Then, according to the local expressions given in §1.1 for the Poisson brackets on $Ab^*(B^*)$ and on T^*B/G, this term is also a Casimir as a function on these spaces. So, the Kaluza-Klein-Kerner Hamiltonian and the pulled-back Hamiltonian used earlier must generate the same equations of motion: Wong's equations.

§2.2 Quagmas (quark-gluon plasmas).

Quark-gluon plasmas, sometimes called **quagmas** are thought to be the state of matter existing immediately after the Big Bang, in heavy-ion collisions provided energies are high enough, and perhaps within neutron stars. A simple introduction to the subject is given in the Scientific American article by McHarris and Rasmussen [1984]. More technical references are the article by Heinz [1983] and the book by Muller [1985].

The quagma equations that we deal with are the non-Abelian versions of the Maxwell-Vlasov equations and are listed below as [YMV1-4]. We call them the Yang-Mills-Vlasov equations. They are the equations for a **collisionless** non-relativistic quagma in the self-consistent field approximation. More realistically, collision terms should be added, and the equations should be made relativistic. See Heinz [1983] for an account of this. Also, spin effects should be added, and of course quantum effects.

The present work was inspired by the paper 'The Hamiltonian Structure of the Maxwell-Vlasov Equations' of Marsden and Weinstein, [1982], in which the Abelian case, i.e. electromagnetic plasmas, were investigated. Their starting point, which will be our starting point, is the phase space $\mathcal{P} = T^*\mathcal{A} \times \mathcal{S}^* = \mathcal{A} \times \mathcal{E} \times \mathcal{S}^*$. A typical element of $\mathcal{P}$ is written (A,Y,f). $A \in \mathcal{A}$ is a connection (Yang-Mills potential) for the trivial bundle $B = \mathbb{R}^3 \times G \to X = \mathbb{R}^3$. Y is an element of $\mathcal{E} = \Gamma(T^*\mathbb{R}^3 \otimes AdB)$. This is the vector space on which the affine space $\mathcal{A}$

of connections is modelled. $E = -Y$ is the "electric" field. Finally $f \in \mathcal{S}^*$ is a quagma density. By this we mean a (generalized) function $f(x, p^{can.}, Q)$ of position x, canonical momentum $p^{can.}$, and color charge Q. <u>To avoid a profusion of indices we now will drop the superscript "can." and denote the canonical momentum by p.</u>

The Poisson structure on $\mathcal{P}$ is obtained by identifying $\mathcal{E}$ with $\mathcal{E}^*$ so that $\mathcal{P}$ is identified with $\mathcal{A} \times \mathcal{E}^* \times \mathcal{S}^* = T^*\mathcal{A} \times \mathcal{S}^*$ and then putting the product Poisson structure on $T^*\mathcal{A} \times \mathcal{S}^*$. The identification of $\mathcal{E}$ with $\mathcal{E}^*$ is done using the L^2-pairing on $\mathcal{E}$ which is induced by the standard metric on $\mathbb{R}^3$ and a fixed bi-invariant metric on $\mathfrak{g}$. In coordinates this pairing is $(E, K) \longmapsto \int E^a{}_i(x) K^i{}_a(x) d^3x$. Here, and throughout, space indices "i" and Lie algebra indices "a" are raised and lowered by their respective metrics.

If one takes the standard (see below) Hamiltonian on this phase space, then Hamilton's equations are the dynamical part of the Yang-Mills Vlasov equations [YMV1-3] below. This was noted by Gibbons, Holm, and Kupershmidt [1982]. However, these authors did not concern themselves with the conservation equation [YMV4] which states that the divergence of the color electric field is the color current. We will show here that this conservation law is the equation $J=0$, where J is the momentum map for the action of the gauge group $\mathcal{G}$ on $\mathcal{P}$.

Marsden and Weinstein [1982] showed this for the Abelian J. In addition to this, their main results are

(i) that the reduced phase space for the $\mathcal{G}$ action can be identified with the space of magnetic fields (curvatures of connections), electric fields, and plasma densities (now as functions of position and velocity), and

(ii) The calculation of the Poisson bracket on this reduced phase space.

To perform the calculation (ii) in the non-Abelian case we are forced to fix a choice of gauge in order to get local coordinates for the quotient space. We opt for the Coulomb gauge.

There is no hope of succeeding with step (i) in the non-Abelian case. This is due to the following basic, but not so well-known result:

In the case where G is non-Abelian, there are gauge **inequivalent** connections on $\mathbb{R}^3 \times G$ with the **same** curvatures and the same holonomy groups (both local and global) on all of $\mathbb{R}^3$.

This was pointed out by Gu [1977] and later by Mostow [1979]. (These authors also show that the connection is in fact determined by its curvature **and** some finite number, depending on G, of its covariant derivatives, under the proviso that the dimension of the holonomy algebra is constant.)

Thus, there is no way to recover the (gauge equivalence class of the) connection, hence no way to even **state** the Yang-Mills equations, given only the E and B fields. We say no way to even **state** the Yang-Mills equations because these equations depend on the covariant derivative, which in turn depends on the connection. (In the

Abelian case, the covariant derivative on $AdB \approx X \times \mathbb{R}$ **is** the exterior derivative, so this connection-dependence is not present.)

Remark: In the Abelian case when the base space is not simply connected one runs into a problem similar to the one pointed out by Gu and Mostow . In this case **global** holonomy data is needed, in addition to the curvature, in order to reconstruct the connection from its curvature. This is true even when the bundle is trivial. For example, $S^1 \times S^1 \to S^1$ supports inequivalent flat connections. In fact such a connection is used to account for the Bohm-Aharanov effect.

The equations.

Before stating the equations, we must clarify the nature of non-Abelian plasma densities. A plasma density is a generalized function f(x,p,Q) on the phase space T*B/G so that p means p^{can}. Its interpretation is

f(x,p,Q)dxdpdQ = amount of color-charged matter in the volume x to x+dx with canonical momenta between p and p+dp and color charge between Q and Q+dQ.

(In the Abelian case Q is **identically** constant, as opposed to **covariantly** constant, so that one may simplify f(x,p,Q) to f(x,p).)

Let

$$F = dA + [A,A]$$

denote the curvature of A,

$$B_i = \epsilon_i{}^{jk} F_{jk}$$

denote the "magnetic field", and

$$D = d + ad_A$$

denote the covariant derivative on Ad(B). D's components are

$$D_i = \partial/\partial x^i + [A_i, \cdot\,]$$

D will be manipulated like the usual "del" of vector calculus. For instance

$$(DxB)_i = \epsilon_i{}^{jk} D_j(B_k)$$

and

$$D{\cdot}E = D_i(E^i)$$

Define the Lie algebra valued current j[f] and density ρ[f] by the

following moments of the plasma density:

$$j_i{}^a(x) = \int p_i Q^a f(x,p,Q)\,dpdQ$$

$$\rho_a(x) = \int Q_a f(x,p,Q)\,dpdQ.$$

Now we can state the

Yang-Mills-Vlasov Equations:

$$d/dt(f) = [v_i \partial/\partial x^i + Q_a(\partial/\partial x^i A^a{}_j) v^j \partial/\partial p_i - Q_d c^d{}_{ab} A^b{}_i v^i\, \partial/\partial Q_a] f$$

$$\text{where } v^i = (p - Q{\cdot}A)^i \qquad \text{[YMV1]}$$

$$d/dt(E) = DxB - j[f] \qquad \text{[YMV2]}$$

$$d/dt(B) = -DxE \qquad \text{[YMV3]}$$

with conservation law

$$(D{\cdot}E)_a = (\rho[f])_a \qquad \text{[YMV4]}\ .$$

(The equation $D\cdot B = 0$ is a consequence of the definition of B and is really the Bianchi identity, so we do not include it as one of the equations.)

The first equation [YMV1] says that f is convected by the flow on T^*B/G of the non-relativistic Hamiltonian vector field X_h, $h = (1/2)v^2$ which was dicussed in the previous section. The other equations [YMV2-4] are equivalent to the space-time Yang-Mills equations $D_\mu F^{\mu\beta} = J^\beta[f]$.

Warnings.

In stating the equations, we have assumed that $A_0 = 0$. This is legitimate, because any space-time connection A is gauge equivalent to one for which A_0 is zero. In fact, A_0 transforms to $g^{-1}A_0g + g^{-1}\partial/\partial t(g)$. Setting this equal to zero and $g = \exp\varphi$ yields the soluble equation $A_0 + \partial/\partial t(\varphi) = 0$. However, if we only want to only allow gauge transformations which go to the identity in the infinite past or future, or are interested in "radiation", then the assumption that $A_0 = 0$ is no longer legitimate.

Other conventions: Holm et al use $F = dA - [A,A]$. They also have $D = d - ad_A$. These differences can mathematically be attributed to using a **left** principal bundle. Mathematicians generally use **right** principal bundles so that $F(X,Y) = dA(X,Y) + [A(X),A(Y)]$. (However, Freed and Uhlenbeck [1985] consider the Hopf fibration $S^7 \rightarrow S^4$ as a

left principal bundle.) Kobayashi-Nomizu write $F = dA + (1/2)[A,A]$ because what they mean by the exterior derivative d is twice what we mean. Arms uses $F = dA + [A,A] = dA + (1/2)A \wedge A$. The meaning of $A \wedge A$ varies from author to author by factors of 2.

Canonical structure.

The space $\mathcal{S}^*$ of plasma densities is to be thought of as the dual of the Lie algebra of the space of Hamiltonian vector fields on T^*B/G. We put the + Lie-Poisson bracket on it. The Poisson bracket on $\mathcal{P} = T^*\mathcal{A} \times \mathcal{S}^*$ is the product Poisson bracket,that is, the sum of the canonical bracket on $T^*\mathcal{A}$ plus the Lie-Poisson bracket on $\mathcal{S}^*$.

Take as Hamiltonian

$$H(A,E,f) = \text{kinetic} + \text{Yang-Mills}$$
$$= (1/2)\int f(x,p,Q)\|p - Q\,A\|^2 dx\,dp\,dQ + (1/2)\int \|E\|^2 + \|B\|^2\, dx.$$

It was noted by Gibbons, Holm, and Kupershmidt [1982] that with this set-up Hamilton's equations are equivalent to the dynamical equations, [YMV1-3]. We will not repeat their calculation.

We now prove our claim that the conservation equation [YMV4] is the equation $J=0$, where J is the momentum map for the action of $\mathcal{G}$ on $\mathcal{P}$. A $g \in \mathcal{G}$ acts according to $g\cdot(A,Y,f) = (gAg^{-1} + gdg^{-1}, Ad_{g^{-1}}{}^{*}E, f\cdot[T^*g^{-1}])$. Recall that $[T^*g^{-1}]$ is the canonical transformation of T^*B/G induced by $g \in \mathcal{G}$ and described in §1.4. The action on the

(A,Y)'s is the cotangent lift of the usual action of $\mathcal{G}$ on $\mathcal{A}$. The momentum map for such a diagonal action is the sum of the momentum maps for each individual factor.

$$J = J_{T^*\mathcal{A}} + J_{\mathcal{S}^*}.$$

It takes values in $\mathfrak{g}^* \simeq \Gamma^*(Ad(B))$. See §1.4 for details concerning this space.

$J_{T^*\mathcal{A}}$ is calculated using the cotangent lift formula for momentum maps. The result is well known (see Arms[1981], Moncrief[1982]):

$$J_{T^*\mathcal{A}}(A,Y) = D\cdot Y = - D\cdot E \quad .$$

$J_{\mathcal{S}^*}$ is calculated using the momentum map j for the action of $\mathcal{G}$ on T*B/G which is given locally by (see §1.4):

$$j(x,p,Q) = Q\delta(x-x').$$

One finds that

$$J_{\mathcal{S}}^*(f) = \langle f, j\rangle = \int f(x,p,Q)Q_a dxdpdQ = \rho[f].$$

Here the brackets "< , >" denote the natural pairing between

distributions f on T*B/G, and functions j on T*B/G with values in the vector space $\mathfrak{g}^*$ of generalized sections of Ad(B). Locally this pairing is the one between distributions on $T^*X \times \mathfrak{g}^*$ and functions on $T^*X \times \mathfrak{g}^*$ with values in the vector space $C^\infty(X,\mathfrak{g}^*)^*$. This first equality is general: if a group acts on a Poisson manifold with momentum map j then $J(f) = \langle f, j\rangle$ is the momentum map for the induced action on the (Lie-Poisson) space of distributions on that Poisson manifold. (This calculation for $J_{\mathcal{S}}^*$ has also been checked directly.)

Putting together these expressions we get the desired result:

$$J(A,Y,f) = -D\cdot E+\rho[f].$$

so that J = 0 is the equation [YMV4].

Reduction using the Couloumb gauge.

Here we give a local formula (local within the space of connections) for the Poisson bracket on the reduced space $\mathcal{P}/\mathcal{G}$. This is achieved by making a choice of gauge in $\mathcal{A}$, that is, a choice of local slice $\tilde{\mathcal{A}} \subset \mathcal{A}$ for the $\mathcal{G}$ action. The choice of gauge we take is called the Coulomb gauge. This gauge is described nicely by Freed and Uhlenbeck [1984, esp. pp.51-58] and Singer [1978]. We review it now in the context of general principal bundles.

Fix $A \in \mathcal{A}$. Then $\mathcal{A} = A + \mathcal{E}$, and

$$D_A: \Gamma(Ad(B)) \longrightarrow \Gamma(T^*X \otimes Ad(B)) = \mathcal{E}.$$

D_A has L^2 adjoint $D_A^*: \mathcal{E} \longrightarrow \Gamma(Ad(B))$. The negative of D_A^* is the operator which we called "D·" above. The $\mathcal{G}$ orbit through A and $\ker D_A^*$ are L^2-orthogonal complements at A. Note that $\Gamma(Ad(B)) = \mathfrak{g}$ is the Lie algebra of the gauge group $\mathcal{G}$. D_A^* can be viewed as the connection one-form for a connection on the "principal bundle" $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$. This connection is called the Coulomb connection.

If g is the metric on the base space X then D_A^* is given locally by

$$D_A^*a = -(1/\sqrt{g})\Sigma_{\alpha,\beta}(\partial_\alpha(g^{\alpha\beta}a_\beta\sqrt{g}) + g^{\alpha\beta}[A_\alpha,a_\beta])$$

For $B = \mathbb{R}^3 \times G$ (or, in normal coordinates at the point in question)

$$D_A^*a = -\partial/\partial x^i a_i - [A_i,a^i]$$

Here a_i is a $\mathfrak{g}$-valued function so that the local expression for $a \in \mathcal{E}$ is $a_i dx^i$, a $\mathfrak{g}$-valued one form on X. (This shows that $-D_A^*$ is the operator we called "D·" above.)

The Coulomb gauge $\tilde{\mathcal{A}}$ through A is defined by

$$\tilde{\mathcal{A}} = A + \tilde{\mathcal{E}}$$

where

$$\tilde{\mathcal{E}} = \{a \in \mathcal{E}: D_A^*a = 0, \text{ and a is "small enough"}\} \subset \mathcal{E}$$

The smallness of a in the definition of $\tilde{\mathcal{E}}$ depends on Sobolev norms of a. $\tilde{\mathcal{A}}$ is a slice for the $\mathcal{G}$ action on $\mathcal{A}$ **if and only if** A is an irreducible connection (has no infinitesimal symmetries). This fact is fundamental to the work of Uhlenbeck, Taubes, and Donaldson which led to Donaldson's celebrated result that there are topological $\mathbb{R}^4$'s which are not diffeomorphic to the standard $\mathbb{R}^4$. The proof that $\tilde{\mathcal{A}}$ is a good slice basically boils down to the fact that the covariant Laplacian"$\Delta_A = D_A^*D_A: \Gamma(Ad(B)) \longrightarrow \Gamma(Ad(B))$ is elliptic.

The fact that $\tilde{\mathcal{A}}$ is a slice means that $\mathcal{U}_A = \tilde{\mathcal{A}}\cdot\mathcal{G}$ is an open neighborhood of $\mathcal{A}$ and that the map

$$(a, \eta) \longmapsto \eta^*(A+a) = A - D\eta\cdot\eta^{-1} + \eta a \eta^{-1}$$

is a diffeomorphism of $\tilde{\mathcal{E}} \times \mathcal{G}$ onto this neighborhood. Moreover, this map provides a local trivialization of the principal $\mathcal{G}$-bundle $\tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}}/\mathcal{G}$, where $\tilde{\mathcal{A}} \subset \mathcal{A}$ denotes the open dense subset of irreducible connections.

This show us that locally $T^*\mathcal{A}/\mathcal{G} \simeq T^*\tilde{\mathcal{E}} \times \mathfrak{g}^* = \tilde{\mathcal{E}} \times \ker D_A^* \times \mathfrak{g}^*$ with the product Poisson bracket (see the local discussion of §1.2).

From this it follows, either by direct calculation, or by invoking the general theory of semi-direct product bundles as in [Montgomery, Marsden, and Ratiu, 1985], that locally $(T^*\mathcal{A} \times \mathcal{S}^*)/\mathcal{G} \simeq \tilde{\mathcal{E}} \times \ker D_A^* \times (\mathfrak{g} \tilde{\times} \mathcal{S})^*$. The second factor is the dual of the semi-direct product Lie algebra $\Gamma(\mathrm{Ad})\, \tilde{\times} \mathcal{S}$, where the action of $\mathcal{G}$ on $\mathcal{S}$ is the one described above. Explicitly, the local bracket is

$$\{F,G\}(a,Y,\varphi,f) = \int [(\delta F/\delta a)(\delta G/\delta Y)-(\delta G/\delta a)(\delta F/\delta Y)]dx$$

$$+ \int \varphi \cdot [\delta F/\delta a, \delta G/\delta Y]dx + \int f\{\delta F/\delta f, \delta G/\delta f\}dx\,dp\,dQ$$

$$+ \int f[\partial_x(\delta F/\delta \varphi)\partial_p(\delta G/\delta f) - \partial_x(\delta G/\delta \varphi)\partial_p(\delta F/\delta f)]dx\,dp\,dQ\ .$$

where

$$(a,Y,\varphi,f) \in \tilde{\mathcal{E}} \times \ker D_A^* \times \mathfrak{g}^* \times \mathcal{S}^*$$

The last four terms are the Lie-Poisson brackets on $(\mathfrak{g}\, \tilde{\times} \mathcal{S})^*$. The final term occurs because the local expression for the infinitesimal action of $\mathfrak{g}$ on $\mathcal{S}$ is given by $\varphi \cdot f = \partial_x \varphi \partial_p f$ where this expression means $\Sigma_i \partial/\partial x^i(\varphi)\partial/\partial p_i(f)$. This concludes our reduction of quagmas.

Remarks.

1. Global bracket formula.

A global bracket formula for the quagma Poisson brackets can be achieved by using the Coulomb connection for the bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. As in §1.2, the Coulomb connection lets us decompose $T^*\mathcal{A}$ into $\mathcal{C} \times \mathfrak{g}^*$ where $\mathcal{C} = \{(A,E) \in T^*\mathcal{A}: D_A^*E = 0\}$ and $\mathfrak{g}^*$ is the dual of the Lie algebra of $\mathcal{G}$. Then $\mathcal{P} = \mathcal{C} \times \mathfrak{g}^* \times \mathcal{S}^*$ and $\mathcal{P}/\mathcal{G} = \mathcal{C} \times_{\mathcal{G}} (\mathfrak{g}^* \times \mathcal{S}^*)$. A global bracket formula for $\mathcal{C} \times_{\mathcal{G}} \mathfrak{g}^*$ is given by PB.1 at the end of §1.2. A similar formula applies to our situation, the only change being that the fiber term of PB.1, which is the Lie-Poisson bracket on $\mathfrak{g}^*$, is replaced by the semi-direct product Lie Poisson bracket on $\mathfrak{g} \tilde{\times} \mathcal{S}^*$. This replacement is detailed by Montgomery, Marsden, and Ratiu [1984].

We did not write down the global formula for the quagma bracket for the following reason. This bracket contains a term which is essentially the canonical bracket on $T^*(\mathcal{A}/\mathcal{G})$. $\mathcal{A}/\mathcal{G}$ is topologically a very complicated space when G is non-Abelian. There is no choice of global coordinates, so although this canonical bracket on $T^*(\mathcal{A}/\mathcal{G})$ can be written down abstractly, it is of little practical value. A similar problem occurs for the curvature term of PB.1.

2. Concerning Ghost Fields.

Cariñena and Ibort [1985] have investigated the Coulomb connection induced symplectic structure on $\mathcal{C}\times\mathfrak{g}^*$ which we mentioned in the above remark. They have shown that the $\mathfrak{g}^*$ factor should be interpreted as the set of ghost fields introduced by Popov in quantizing Yang-Mills. They also have shown that the B.R.S. transformation, another tool in the quantization of Yang-Mills, is the infinitesimal action of $\mathfrak{g}$ on $\mathcal{C}\times\mathfrak{g}^*$ corresponding to the diagonal action of $\mathcal{G}$. As noted in §1.4, the momentum map for this action is the projection onto the $\mathfrak{g}^*$ factor.

3.Regarding irreducibility.

The flat connection is *not* irreducible (in fact it has the maximum possible amount of symmetry). So one is not able to perform the reduction as above in a neighborhood of this connection. The instanton connections for the Hopf bundle $S^7 \to S^4$ are irreducible. It is not clear though how these would apply to a physical quagma.

From general considerations (see Marsden [1981]), one expects that symplectic reduction in the neighborhood of a reducible connection should result in a stratified symplectic manifold, with the strata corresponding to the various symmetry types. The specific case of $J_{T^*\mathcal{Q}}^{-1}(0)/\mathcal{G} \subset T^*\mathcal{Q}/\mathcal{G}$ was worked out by Arms [1979], [1980] and [1981]. We expect that the quagma situation is very similar to this one.

§2.3 Water Drops.

In this section we state the Hamiltonian structure for the flow of an incompressible fluid with free boundary and surface tension, eg. a water drop. There are two versions of the Poisson bracket, one corresponding to T^*B/G and one to $Ad^*(B^*)$ Both generalize the canonical brackets which Zakharov [1968] found in the irrotational case. For proofs and more details of the material stated here, see Lewis, Marsden, Montgomery, and Ratiu [1985]. For applications of this work to the stability of rotating water drops, see Lewis, Marsden, and Ratiu [1985].

The configuration space $\mathcal{C}$ for an incompressible water drop is the manifold $Emb_{vol}(B,\mathbb{R}^3)$ of volume preserving embeddings of a the three-dimensional reference ball B into $\mathbb{R}^3$. For definiteness, assume that the ball is the standard ball so has volume $^4/_3\pi$. The boundary of B is the unit sphere S^2. If $\eta \in \mathcal{C}$ then $\eta: B \to \mathbb{R}^3$ and $\eta(S^2) = \Sigma$ is the free boundary of the water drop. The phase space for the water drop is the cotangent bundle $T^*\mathcal{C}$ of $\mathcal{C}$. This is the fluid mechanicians' space of Lagrangian variables (η,μ). So η assigns to each reference point $X \in B$ a spatial point $x = \eta(X)$ and $\mu \in T_\eta{}^*\mathcal{C}$ is a divergence-free vector-field over η: that is, μ assigns to each reference point $X \in B$ a momentum (covector) based at the spatial point $x = \eta(X)$. (As always, we identify vectors and covectors on $\mathbb{R}^3$.) The corresponding energy of the drop is

$$H((\eta,\mu)) = {}^1/_2\int_B |\mu|^2 d^3X + \tau\int_\Sigma dA \qquad [3.1]$$

where the density of the drop is assumed to be 1, and its coefficient of surface tension, τ is assumed to be constant over the surface of the boundary.

The energy is invariant under the right action of the particle relabelling group $G = \mathrm{Diff}_{vol}(B)$, the group of volume preserving diffeomorphisms of the ball. Here a $\psi \in G$ acts on the pair (η,μ) by right composition on each factor. This is the cotangent lift of the action $\eta \longmapsto \eta\circ\psi$ on $\mathcal{C}$. So, the dynamics descends to $T^*\mathcal{C}/G$. This quotient space can be identified with pairs (Σ,v) where Σ is an embedded S^2 in $\mathbb{R}^3$ (forgetting about the embedding) which bounds a region $D = D_\Sigma$ of volume ${}^4/_3\pi$, and where v is a divergence-free vector field on D_Σ. The identification map is $[\eta,\mu] \longmapsto (\Sigma = \eta(S^2), v = \mu\circ\eta^{-1})$. Using this identification map and the canonical brackets on $T^*\mathcal{C}$ one calculates that the reduced Poisson bracket on the (Σ,v)'s is

$$\{F, G\}(\Sigma,v) = \int_D \langle\omega, \delta F/\delta v \times \delta G/\delta v\rangle d^3x + \int_\Sigma [(\delta F/\delta\Sigma)(\delta G/\delta\varphi) - (\delta G/\delta\Sigma)(\delta F/\delta\varphi)]dA \qquad [3.2]$$

where

$$\omega = \nabla \times v$$

is the vorticity. The definition of the functional derivatives is as follows. $\delta F/\delta v$ is a divergence free vector field on D_Σ which is defined by

$$DF\cdot(0,\delta v) = \int_D \langle\delta F/\delta v, \delta v\rangle d^3x.$$

The interpretation of $\delta F/\delta\Sigma$ takes some care. Denote the space of free boundaries, i.e. Σ's, by $\mathcal{I}m$ (for images). Think of $\delta\Sigma \in T_\Sigma\mathcal{I}m$ as a normal variation of Σ, that is, a function on Σ. Since the volume of the D_Σ's must remain constant, we have that $\int_\Sigma (\delta\Sigma)dA = 0$. Then

$$DF\cdot(\delta\Sigma,0) = \int_D (\delta F/\delta\Sigma)\delta\Sigma dA.$$

Finally, given a function f on Σ whose integral over Σ is zero, let $N(f)$ denote the harmonic function ψ on D_Σ obtained by solving the Neumann problem

$$\Delta\psi = 0 \text{ on } D_\Sigma$$

$$\partial\psi/\partial\upsilon = f \text{ on } \Sigma .$$

Here $\upsilon = \upsilon_\Sigma$ is the normal vector to Σ. Then

$$\delta F/\delta\varphi = N(\delta F/\delta v \text{ restricted to } \Sigma) \qquad [3.3]$$

One checks that $\delta F/\delta\varphi$ is the variation of F with respect to potential variations. Note that in the irrotational case, $\omega = 0$, so the bracket is given by the second term alone, which is exactly Zakharov's bracket.

The energy in the reduced variables is

$$H(\Sigma,v) = {}^1/_2\int_D |v|^2\, d^3x + \tau\int_\Sigma dA .$$

One may check directly that with the Poisson brackets [3.2], this Hamiltonian generates the correct equations of motion, namely

$$\partial v/\partial t + (v\cdot\nabla)v = -\nabla p$$

$$\delta\Sigma/\delta t = v\cdot\upsilon_\Sigma$$

$$\operatorname{div} v = 0$$

$$p = \tau\kappa \text{ on the boundary } \Sigma$$

where κ is the mean curvature of Σ. (As usual in the incompressible fluids, the occurrence of the pressure p may be seen as a mechanism forcing $\partial v/\partial t$ to remain divergence free, rather than as another variable to be soved for in addition to v and Σ.)

We have seen that the (Σ,v) brackets [3.2] correspond to the Poisson brackets on $T^*\mathfrak{C}/G$. To obtain the co-adjoint bundle description, we begin by noting that $\mathfrak{C} \longrightarrow \mathfrak{Sm}$ is a principal G-bundle where the projection is $\eta \longmapsto \Sigma = \eta(S^2)$. This principal bundle has a natural connection, the "Neumann connection". The horizontal lift for this connection is

$$h_\eta(\delta\Sigma) = \nabla(N(\delta\Sigma))\circ\eta^{-1} .$$

The co-adjoint bundle $Ad^*(\mathfrak{C}^*)$ is identifiable with triples (Σ,φ,w) where φ is a harmonic function on D_Σ and w is a divergence free vector field **parallel** to Σ.

The Weyl-Hodge theory states that any divergence free vector field on D_Σ can be decomposed uniquely as

$$v = w + \nabla\varphi$$

The map $(\Sigma,\varphi,w) \longmapsto (\Sigma,v)$ *is* the minimal coupling procedure, $[\Phi(N)]$, for the Neumann connection. One finds that the brackets induced on $Ad^*(\mathcal{C}^*)$ are

$$\{F,G\}(\Sigma,\varphi,w) = \int_\Sigma [(\delta F/\delta\Sigma)(\delta G/\delta\varphi) - (\delta G/\delta\Sigma)(\delta F/\delta\varphi)]dA$$

$$+ \int_\Sigma \langle\omega, \upsilon_\Sigma \times [\nabla(N(\delta F/\delta\varphi)) \times \nabla(N(\delta G/\delta\varphi))]\rangle\, dA$$

$$+ \int_D \langle\omega, [\delta F/\delta w + \nabla(N(\delta F/\delta\varphi))] \times [\delta G/\delta w + \nabla(N(\delta G/\delta\varphi))]\rangle d^3x$$

These terms correspond respectively to the canonical brackets on $T^*\mathcal{S}m$, the curvature term, and the Lie-Poisson term on the fibre, just as in the intrinsic formula PB.1 of §1/2. Note that in the fixed boundary case only the Lie-Poisson term is present. This is the fixed boundary bracket found by Marsden and Weinstein [1983]. And in the irrotational case the bracket again reduces to Zakharov's.

Appendix: Dual pairs and some facts concerning reduction.

Let S be a symplectic manifold with symplectic form ω_S. Suppose the Lie group G acts canonically on the right on S with equivariant momentum map $J:S \to \mathfrak{g}^*$, and suppose that $\mu \in \mathfrak{g}^*$ is a regular value of J. (This is equivalent to the G action being locally free near $J^{-1}(\mu)$. For the definition of momentum maps and some basic facts concerning them see Marsden [1981].) J is then a Poisson map from S to $\mathfrak{g}_-^*$ ($\mathfrak{g}^*$ with its minus Lie-Poisson structure). Let $\mathcal{O}$ denote the co-adjoint orbit through μ and $\omega_{\mathcal{O}}^-$ its minus orbit symplectic structure. The quotient manifold S/G is a Poisson manifold. (See Weinstein [1983] for a nice exposition on Poisson manifolds.) Let $x \in J^{-1}(\mu) \subset J^{-1}(\mathcal{O})$, let $[x]$ be its projection by $\pi:S \to S/G$, let L denote the symplectic leaf throgh $[x]$, and let ω_L denote the symplectic form on this leaf. Let i denote the inclusion $J^{-1}(\mathcal{O}) \hookrightarrow S$. The Marle-KKS formula states:

$$i^*\omega_S = \pi^*\omega_L + J^*\omega_{\mathcal{O}}^- . \qquad \text{[MKKS]}$$

This formula was used in the proof of theorem 1 of §1.3. We will prove this formula as a corollary to a more general formula concerning dual pairs.

Recall (Weinstein [1985]) that a **dual pair** is a pair (π_1,π_2) of Poisson maps $P_1 \xleftarrow{\pi_1} S \xrightarrow{\pi_2} P_2$, where S is symplectic, whose corresponding function groups $\mathfrak{F}_1 = \pi_1{}^*C^\infty(P_1)$ and $\mathfrak{F}_2 = \pi_2{}^*C^\infty(P_2)$ are **polar** . (Polar function groups are Lie subalgebras of $C^\infty(S)$ which are each other's annihilators under Poisson bracket.). A dual pair is called **full** if π_1 and π_2 are both submersions. In the above paragraph $S/G \xleftarrow{\Pi} S \xrightarrow{J} \mathfrak{g}^*$ is an example of a full dual pair.

In the following we suppose that $P_1 \xleftarrow{\Pi_1} S \xrightarrow{\Pi_2} P_2$ is a full dual pair. Let $y \in S$ and let $B_y: T^*_yS \rightarrow T_yS$ denote the Poisson structure on S. By abuse of notation, we will write $B_y\mathfrak{F}_1 \subset T_yS$ for $B_y d\mathfrak{F}_1$. Let $L_1(y)$ denote the symplectic leaf through $\pi_1(y)$.

Lemma.

$$\ker T_y\pi_1 = B_y\mathfrak{F}_2, \ \ker T_y\pi_2 = B_y\mathfrak{F}_1 . \qquad [A.1.1]$$

$$T_y\pi_1{}^{-1}(L_1(y)) = B_y\mathfrak{F}_1 + B_y\mathfrak{F}_2 . \qquad [A.1.2]$$

Proof. To prove the first identity, recall that $B_y\mathfrak{F}_2 = B_y\mathfrak{F}_1{}^\perp$ where $\perp$ denotes ω-orthogonal complement. So $v \in B_y\mathfrak{F}_2 \Leftrightarrow \omega(v, B_y d\pi_1{}^*F) = 0$ for all F on $P_1 \Leftrightarrow dF \cdot T\pi_1 v = 0$ for all F on $P_1 \Leftrightarrow v \in \ker T\pi_1$. This proves that $\ker T_y\pi_1 = B_y\mathfrak{F}_2$ and the proof that $\ker T_y\pi_2 = B_y\mathfrak{F}_1$ is the same. To prove identity [A.1.2], note that

$T\pi_1 B_y\mathfrak{F}_1 = B^{(1)}{}_{\pi_1 y}C^\infty(P_1) = T_yL_1(y)$, where $B^{(1)}{}_y$ is the Poisson tesnor on P_1. Thus $T_y(\pi_1{}^{-1}(L_1)) = T_y\pi_1{}^{-1}(T_yL_1(y)) = B_y\mathfrak{F}_1 + \ker T_y\pi_1 = B_y\mathfrak{F}_1 + B_y\mathfrak{F}_2$.

Theorem

Suppose that $P_1 \xleftarrow{\Pi_1} S \xrightarrow{\Pi_2} P_2$ is a full dual pair. Let $x \in S$, $x_1 = \pi_1(x)$, and $x_2 = \pi_2(x)$. Let L_i denote the symplectic leaf through x_i, and ω_i be the symplectic form on this leaf. Assume (for simplicity) that $\pi_i{}^{-1}(x_i)$ and $\pi_i{}^{-1}(L_i)$ are connected. Then:

1. $\pi_1(\pi_2{}^{-1}(x_2)) = L_1$. $\pi_2(\pi_1{}^{-1}(x_1)) = L_2$.

2. $\pi_1{}^{-1}(L_1) = \pi_2{}^{-1}(L_2)$.

3. Let $i: \pi_1{}^{-1}(L_1) \hookrightarrow S$ be the inclusion. Then

$$i^*\omega_S = \pi_1{}^*\omega_1 + \pi_2{}^*\omega_2. \qquad [A.1.3]$$

4. The symplectic manifold $\pi_1{}^{-1}(L_1)/\ker(i^*\omega_S)$ is a covering symplectic manifold for $L_1 x L_2$.

Corollary 1.

Suppose that $(\pi_1,\pi_2) = (\pi,J)$ as in the first paragraph. Then the leaves of S/G are the submanifolds $J^{-1}(\Theta)/G$ (statement 2 of the theorem) and the Marle-KKS formula [MKKS] holds (statement 3 of the theorem).

Remark. The symplectic manifold of part 4 of the theorem is the reduced space as defined by Kazhdan, Kostant and Sternberg [1978]. In the case $(\pi_1,\pi_2) = (\pi,J)$ part 4 says that this manifold is isomorphic, up to coverings, to $J^{-1}(\Theta)/G \times \Theta$, which is a result of Kazhdan, Kostant and Sternberg [1978].

Corollary 2.

$B_x\mathfrak{I}_1 \cap B_x\mathfrak{I}_2 = \ker T\pi_1 \cap \ker T\pi_2 = \ker(i^*\omega_S)$. These vector spaces can be identified with the fiber of the co-normal bundle of L_1 at x_1 (or of L_2 at x_2) i.e to the space of Casimirs at x_1.

Proof. This follows directly from formula [A.1.3] of the theorem and the second inclusion of the lemma. The identification of $B_x\mathfrak{I}_1 \cap B_x\mathfrak{I}_2$ with the co-normal bundle $N^*_{x_1}L_1 = \{\alpha \in T^*_{x_1}P_1 : \alpha$ annihilates $T_{x_1}L_1\}$ is $\alpha \longmapsto B_x\pi_1{}^*\alpha$.

<u>**Proof of Theorem.**</u>

Proof of (1). Let $y \in \pi_2{}^{-1}(x_2)$ and let γ be a path in $\pi_2{}^{-1}(x_2)$ joining x to y. From the identity [A.1.1] we have $d\gamma(t)/dt = B_{\gamma(t)}d\pi_1{}^*F_t$. Let $\gamma_1(t) = \pi_1\gamma(t)$ denote the projected curve. Then $d\gamma_1(t)/dt = T\pi_1 \cdot B_{\gamma(t)}d\pi_1{}^*F_t = B^{(1)}{}_{\gamma_1(t)}dF_t$, and so $\pi_1(y)$ is in the same symplectic leaf as x_1. The same argument, with the roles of the indices 1 and 2 switched proves the second statement of (1).

Proof of (2). Let $y \in \pi_1{}^{-1}(L_1)$ and let γ denote a differentiable path connecting x to y. By the above identities we have $d\gamma(t)/dt = B_{\gamma(t)}d\pi_2{}^*F_t + v_2$ where $v_2 \in \ker T\pi_2$. Then $d\pi_2\gamma(t)/dt = B^{(2)}{}_{\pi_2\gamma(t)}dF_t$, thus $\pi_2(y)$ is in L_2. This implies that $\pi_2(\pi_1{}^{-1}(L_1)) \subset L_2$, and so $\pi_1{}^{-1}(L_1) \subset \pi_2{}^{-1}(L_2)$. A symmetric argument shows that $\pi_1{}^{-1}(L_1) \supset \pi_2{}^{-1}(L_2)$.

Proof of (3). Consider functions $f_1, g_1 \in \mathfrak{I}_1$ and $f_2, g_2 \in \mathfrak{I}_2$. Write X_f for the Hamiltonian vector field corresponding to the function f: $X_f(x) = B_x df$. Identity [A.1.2] of the lemma states that vectors of the form $X_{f_1}(x) + X_{f_2}(x)$ span $T_x\pi_1{}^{-1}(L_1)$. Thus, to prove the validity of formula [A.1.3] it suffices, by the bilinearity of both sides, to verify that both two-forms give the same result when evaluated on pairs of vectors of the form (X_{f_1},X_{g_1}), (X_{f_2},X_{g_2}), or (X_{f_1},X_{g_2}). To prove equality for pairs of the first form, write $f_1 = \pi_1{}^*F_1$ and similarly for g_1. Then $\omega(X_{f_1},X_{g_1}) = \{f_1,g_1\}_S = \{F_1,G_1\}_{P_1} = \omega_1(X_{F_1},X_{G_1}) = \omega_1(T\pi_1X_{f_1},T\pi_1X_{g_1}) = \pi_1{}^*\omega_1(X_{f_1},X_{g_1})$. The same argument works for pairs of the second form. For pairs of the third form, both two-forms yield zero: $\omega(X_{f_1},X_{g_2}) = 0$ because, by definition of dual pair, X_{f_1} is in the ω-orthogonal complement to X_{g_2}. $\pi_1{}^*\omega_1(X_{f_1},X_{g_2}) = 0$ since $X_{g_2} \in \ker T\pi_1$ by identity [A.1.2], and similarly $\pi_2{}^*\omega_2(X_{f_1},X_{g_2}) = 0$.

Proof of (4). Consider the map $\pi(s) = (\pi_1(s),\pi_2(s))$ from $\pi_1{}^{-1}(L_1)$ to $L_1 \times L_2$. From the equality $\ker(i^*\omega_S) = \ker T\pi_1 \cap \ker T\pi_2$ it follows that π induces a map $[\pi]:\pi_1{}^{-1}(L_1)/\ker(i^*\omega_S) \longrightarrow L_1 \times L_2$, and that $T[\pi]$ is injective. Since our dual pair is full, $T[\pi]$ is also surjective, so the inverse function theorem implies that $[\pi]$ is a local diffeomorphism, and hence a covering map. It is clear that $[\pi]$ is symplectic. ●

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Note added in proof

The proof in §7 of the validity of property 1, $DI = 0$, is incorrect for the non-Abelian case. (This is because $\langle I \rangle \neq I$.) Property 1 still holds in this case, and can be proved as follows. Note that $[\langle v \rangle, \lambda] = 0$, and apply this Lie bracket to an arbitrary function f to conclude that $D_V I^\lambda$ is constant. The proof then continues as before.

details: $[\langle v \rangle, \lambda] = 0$ for $\lambda \in g$ because $\langle v \rangle$ is G-invariant.

$$[\langle v \rangle, \lambda][f] = \langle v \rangle \lambda[f] - \lambda \langle v \rangle [f]$$
$$= D_V\{f, I^\lambda\} - \{D_V f, I^\lambda\}$$
$$= \{f, D_V I^\lambda\}$$

where in the last line we used the fact that D_V is a derivation with respect to Poisson brackets. Since this expression is zero for all f, we have that $D_V I^\lambda$ is constant on the fibers $P \times \{m\}$. Thus $D_V I^\lambda = \langle D_V I^\lambda \rangle$. Now as shown before, $\langle D_V I^\lambda \rangle = \langle d_M I^\lambda \rangle$.

Errata for 'The Bundle Picture in Mechanics'

The equation on the bottom of p. 109 should read

$$J_S^*(f) = \langle f, j \rangle = \int f(x,p,Q)\, Q\, dp\, dQ = \rho(f).$$

The next-to-last sentence on p. 113 should read:
"The final term occurs because the local expression for the infinitesimal action of $\mathfrak{g}$ on S is given by

$$\varphi \cdot f = Q \cdot (\partial_x \varphi \cdot \partial_p f + [\varphi, \partial_Q f])$$

where $\partial_x \varphi \cdot \partial_p f = \sum \partial\varphi/\partial x^i \, \partial f/\partial p_i$."

The bracket on p. 113 should read:

$$\{F,G\}(a,Y,\varphi,f) = \int \frac{\delta F}{\delta a}\frac{\delta G}{\delta Y} - \frac{\delta G}{\delta a}\frac{\delta F}{\delta Y}\, dx$$

$$+ \int \varphi\left[\frac{\delta F}{\delta\varphi}, \frac{\delta G}{\delta\varphi}\right] dx + \int f \left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} dx\, dp\, dQ$$

$$+ \int f Q \cdot \left\{ \partial_x \frac{\delta F}{\delta\varphi}\, \partial_p \frac{\delta G}{\delta f} + \left[\frac{\delta F}{\delta\varphi}, \partial_Q \frac{\delta G}{\delta f}\right] - \partial_x \frac{\delta G}{\delta\varphi}\, \partial_p \frac{\delta F}{\delta f} - \left[\frac{\delta G}{\delta\varphi}, \partial_Q \frac{\delta F}{\delta f}\right]\right\}$$

$$\times\, dx\, dp\, dQ$$