

Rotating Eights I : the three Γ_i families

Alain Chenciner^{1,2}, Jacques Féjoz^{1,3} and Richard Montgomery⁴

¹ ASD, IMCCE (UMR 8028 du CNRS), Observatoire de Paris

² Département de Mathématiques, Université D. Diderot

³ Institut de Mathématiques (UMR 7586 du CNRS), Université P. & M. Curie

⁴ Department of Mathematics, University of California Santa Cruz

E-mail: chencine@imcce.fr, fejoz@math.jussieu.fr, rmont@math.ucsc.edu

Abstract. We show that three families of relative periodic solutions bifurcate out of the Eight solution of the equal-mass 3-body problem : the planar Hénon family, the spatial Marchal P_{12} family and a new spatial family. The Eight, considered as a spatial curve, is invariant under the action of the 24-element group $D_6 \times Z_2$. The three families correspond to symmetry breakings where the invariance group becomes isomorphic to D_6 , the three D_6 's being embedded in the larger group in different ways. The proof of the existence of these three families relies on writing down the action integral in a rotating frame, viewing the angular velocity of the frame as a parameter, exploiting the invariance of the action under a group action which acts on the angular velocities as well as the curves and, finally, checking numerically the non-degeneracy of the Eight. Pictures and numerical evidence of the three families are presented at the end.

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1. Results: three families of rotating Eights.

We consider the equal mass Newtonian three-body problem in 3-space :

$$\frac{d^2x_i}{dt^2} = \frac{\partial U}{\partial x_i}(x_1, x_2, x_3), \quad U(x_1, x_2, x_3) = \sum_{i < j} \frac{1}{\|x_i - x_j\|}, \quad x_i \in \mathbb{R}^3.$$

The Eight (figure 1) is a recently discovered periodic planar solution to this problem (see [Mo, CM]). Due to the homogeneity of the potential function U , if $x(t)$ is a solution then so is $\mu^{-2/3}x(\mu t)$ for every $\mu > 0$, in particular, to any periodic solution there corresponds a rescaled periodic solution with a given period T . Hence from now on, we shall stick to a given period T .

The Eight has a large symmetry group Γ (see section 4.3). It is *choreographic*, meaning that the three masses of the Eight travel along a single curve in space. This space curve is a planar figure Eight, with three obvious symmetry axes, two in the plane of the Eight and the third orthogonal to that plane.

We can rotate (in space) any solution to the three-body problem to obtain a new solution. We can also translate time. As a consequence of these symmetries, the linearization of the return map (= time T map of the flow) associated to a periodic solution of period T will have 1 as an eigenvalue. Provided the solution is not a relative equilibrium (orbit of the rotation group) then the multiplicity of the eigenvalue will be

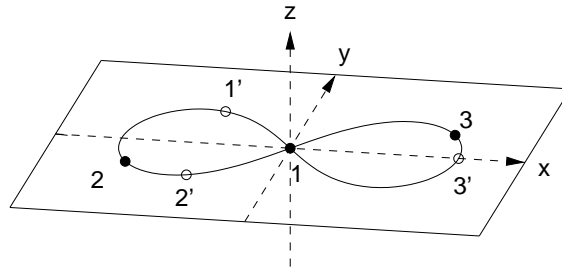


Figure 1. The Eight

at least 4; 3 for the rotation group and 1 for time translation. (We fix the center of mass to be zero so translations are not allowed.) We shall call that solution *Poincaré non-degenerate* if the dimension of the eigenspace associated to the eigenvalue 1 is exactly 4. Because the flow of the n -body problem is Hamiltonian and the angular momentum of the Eight is equal to 0, the algebraic multiplicity of the eigenvalue 1 is at least 8 (and in fact equal to 8, see section 4) but we shall show that the Eight is Poincaré non-degenerate by checking numerically its Jordan block structure. This non-degeneracy is consistent with the fact, proved in [K] using interval arithmetic, that, among T -periodic curves, the Eight is isolated modulo rotations, translations, and scaling. It completes the numerical proof of stability of the Eight after reduction given in [S].

A solution is called *relatively T -periodic* if it is periodic up to a rigid rotation of the three bodies or, equivalently, if there is a rotation R such that $x(T) = Rx(0)$ and $\dot{x}(T) = R\dot{x}(0)$ (where R acts diagonally on the positions and velocities of the bodies). Upon writing $R = \exp(\omega T)$ for an angular velocity ω , the solution becomes T -periodic when viewed with respect to a rotating frame which is rotating with constant angular velocity ω .

The main goal of this paper is to prove the following result.

Theorem 1 (Existence of the Γ_i families) *Under the hypothesis, numerically verified, that the Eight is Poincaré non-degenerate, there are three families $\{x_\lambda^i\}$, $i = 1, 2, 3$, $\lambda \in \mathbb{R}$, $|\lambda| < \lambda_0$, of relatively T -periodic solutions bifurcating out of the Eight, one family for each symmetry axis e_1, e_2, e_3 of the Eight. Viewed in the frame rotating around the axis e_i with angular speed λ , the solution x_λ^i is T -periodic, depends analytically on λ , and has symmetry group Γ_i described in section 4.3.*

Two of these families were known. One was discovered by Michel Hénon [H2] using the numerical methods of [H1], another by Christian Marchal [Ma2]. The third family is new. Pictures of the three families in the rotating frame, obtained numerically, are given at the end of the paper.

About the unicity of the Γ_i families. After seeing a former version of this paper, David Chillingworth showed us how it would be possible to check using [CH] that no other families of relative periodic solutions bifurcate from the Eight. Carles Simó also told us that uniqueness can be proved by showing that some function of the coefficients of the monodromy matrix (see section 4.2) is non zero. Furthermore he checked this numerically.

Outline of proof. Any relatively periodic solution is periodic in a frame which rotates with an appropriate angular velocity ω . Treat ω as a bifurcation parameter. The problem becomes a problem in the theory of bifurcations with symmetry. There is a parameterized action principle A_ω whose non-collision critical points are solutions periodic in the frame rotating with constant angular velocity ω . The Eight is a critical point for A_0 . Applying the symmetries of rotation and time translation to the Eight yields a 4-dimensional critical submanifold \mathcal{E} of Eights for A_0 . The idea is to turn on ω and simultaneously move normally to \mathcal{E} so as to look for critical points to A_ω in the space normal to \mathcal{E} at a given Eight. This is the Lyapunov-Schmidt reduction (section 4.1) and it reduces the search for nearby relatively periodic solutions to the search for critical points of a certain function on \mathcal{E} . Using group theory, in section 4.5 we reduce the latter critical point problem to a non-linear eigenvalue problem. In section 5 we solve this non-linear eigenvalue problem explicitly (again using group theory) under the assumption that ω is parallel to any one of the three symmetry axes of the Eight (section 5). Hence the three families. The Lyapunov-Schmidt method requires the non-degeneracy of the Hessian of A_0 in directions orthogonal to \mathcal{E} . We show in lemma 2 of section 4.1 that this non-degeneracy is equivalent to the non-degeneracy of the Poincaré map associated to the Eight. We verify this latter non-degeneracy numerically in section 4.4.

2. The Lagrangian and the action in a rotating frame.

The configuration space \mathcal{X} for the equal mass spatial three-body problem consists of triples $x = (x_1, x_2, x_3)$ of vectors $x_i \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 = 0$. If $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is an orthogonal transformation then we write Rx for (Rx_1, Rx_2, Rx_3) . To formulate dynamics in a rotating frame, we take $R = R(t)$ a family of rotations depending on time, and use as variable $\xi(t)$ where

$$x(t) = R(t)\xi(t).$$

We are only interested in frames rotating with constant angular velocity ω , in which case

$$R(t) = \exp(t\omega).$$

Some words are in order regarding the notation $\exp(t\omega)$. Throughout this paper we will be using the standard identifications of \mathbb{R}^3 with the Lie algebra $so(3)$ of skew-symmetric linear maps of \mathbb{R}^3 and with the spaces $\Lambda^2\mathbb{R}^{3*}$ of exterior 2-forms and $\Lambda^2\mathbb{R}^3$ of 2-vectors. Under the first identification $\omega \in \mathbb{R}^3$ defines the skew-symmetric linear operator $v \mapsto \omega \times v$. We can exponentiate the latter operator, obtaining the rotation “ $R(t) = \exp(t\omega)$ ”. Under the second identification, ω defines the 2-form $(v, w) \mapsto \langle \omega, v \times w \rangle$. Under these identifications, if (e_1, e_2, e_3) is an oriented basis for \mathbb{R}^3 then e_3 represents the rotation of angle $+\pi/2$ in the (e_1, e_2) plane which in turn is identified with the bivector $e_1 \wedge e_2$. *It is fundamental in the sequel to keep in mind that ω is a bivector, and so transforms as a bivector, not a vector.*

The Lagrangian L_ω governing dynamics in the moving frame is

$$L_\omega(\xi, \dot{\xi}) = \frac{1}{2} \|\omega \times \xi + \dot{\xi}\|^2 + U(\xi) = L_0(\xi, \dot{\xi}) + \frac{1}{2} \|\omega \times \xi\|^2 + \langle \omega \times \xi, \dot{\xi} \rangle.$$

Here we have used the invariance of U , the negative of the potential, under rotations: $U(R\xi) = U(\xi)$. This Lagrangian can be rewritten as

$$L_\omega(\xi, \dot{\xi}) = L_0(\xi, \dot{\xi}) + \frac{1}{2} \langle \omega, I(\xi)\omega \rangle + \langle \omega, J(\xi, \dot{\xi}) \rangle,$$

where I is the inertia matrix of the solid defined by the configuration ξ and J is the angular momentum of the phase element $(\xi, \dot{\xi})$. Note that L_0 is the Lagrangian for the 3-body problem in an inertial frame. The action in the rotating frame is given by the formula

$$A_\omega(\xi) = A_0(\xi) + \frac{1}{2} \langle \omega, \mathbb{I}(\xi)\omega \rangle + \langle \omega, \mathbb{J}(\xi) \rangle,$$

where \mathbb{I} and \mathbb{J} are the integrated versions of I and J . We take the domain of the action A_ω to be the loop space $H^1 = H^1(S^1, \mathcal{X})$ consisting of those maps from the time circle $S^1 = \mathbb{R}/T\mathbb{Z}$ to the configuration space \mathcal{X} which, together with their derivative, are square integrable. The choice of H^1 does allow for collisions. However, because H^1 is contained in the space C^0 of continuous paths, and because all our analysis is local in an H^1 -neighborhood of the Eight, collisions will be avoided, at least for small values of the bifurcation parameter ω .

3. Symmetries and symmetry breakings

We introduce a group G which acts naturally on the space of relatively T -periodic solutions when the three masses are equal. For the notations, we follow Ferrario and Terracini [Fe].

3.1. Action on loops

The group $G = O(3) \times O(2) \times S_3$ acts on the loop space H^1 in the following way. The factor $O(3)$ acts diagonally on the configuration space \mathcal{X} . The factor $O(2)$ acts on the time circle $\mathbb{R}/T\mathbb{Z}$. The third factor S_3 –the symmetric group on three letters– acts on the three indices 1, 2, 3 which label the three bodies. G acts on H^1 according to

$$(\rho, \tau, \sigma) \cdot (\xi_1, \xi_2, \xi_3)(t) = (\rho\xi_{\sigma^{-1}(1)}(\tau^{-1}t), \rho\xi_{\sigma^{-1}(2)}(\tau^{-1}t), \rho\xi_{\sigma^{-1}(3)}(\tau^{-1}t)).$$

3.2. Action on bivectors

We now define an action of G on the space $\Lambda^2\mathbb{R}^3$ of bivectors ω , such that

$$A_{g \cdot \omega}(g\xi) = A_\omega(\xi)$$

for $g = (\rho, \tau, \sigma) \in G$. The permutation group factor $\sigma \in S_3$ acts trivially on bivectors. The group $O(2)$ of time isometries acts according to the sign homomorphism: $\tau \cdot \omega = \det(\tau)\omega$. Thus any orientation-preserving isometry of the time circle acts as the identity on ω while an orientation-reversing isometry reverses the sign of ω . This reversal is seen to follow either from the definition of angular velocity as a time derivative, or by its duality with angular momentum and the formula for angular momenta in terms of $\xi \wedge \dot{\xi}$. Finally the linear isometry group $O(3)$ of \mathbb{R}^3 acts in the standard way by push-forward. If we identify $\Lambda^2\mathbb{R}^3$ with \mathbb{R}^3 as above, then $\rho \in O(3)$ sends $\omega \in \mathbb{R}^3$ to $\rho \cdot \omega = \det(\rho)\rho\omega$.

3.3. Space of Eights

Let x_8 be the figure Eight solution. Write $\mathcal{E} = Gx_8$ for the orbit of the Eight under the G -action. By definition, \mathcal{E} is the space of Eights. It is a four-dimensional submanifold of H^1 isomorphic to G/Γ where the finite group $\Gamma \subset G$ is the isotropy group of the Eight (see section 4.3 for more on Γ). The submanifold \mathcal{E} admits a tubular neighborhood \mathcal{T} in the full loop space H^1 (use normal projection), which allows us to introduce coordinates (η, ζ) in \mathcal{T} , with $\eta \in \mathcal{E}$ and $\zeta \perp T_\eta \mathcal{E}$, and $\eta + \zeta$ sweeping out the neighborhood \mathcal{T} . To stress the dependence of the action on these variables, we will write

$$A_\omega(\eta + \zeta) = A(\eta, \zeta, \omega).$$

4. The Liapunov-Schmidt Reduction

4.1. Initial Reduction

The Euler-Lagrange equation $dA_\omega(\eta + \zeta) = 0$ is equivalent to the two equations

$$\frac{\partial A}{\partial \eta}(\eta, \zeta, \omega) = 0, \quad \frac{\partial A}{\partial \zeta}(\eta, \zeta, \omega) = 0. \quad (1)$$

These equations are identically satisfied on the space of Eights ($\zeta = 0$) for $\omega = 0$. According to the implicit function theorem, we can solve the second equation for a

$$\zeta = \zeta(\eta, \omega) \text{ for } |\omega| < C$$

defined in a neighborhood $|\omega| < C$ of $\omega = 0$ provided that $\frac{\partial^2 A}{\partial \zeta^2}(\eta, 0, 0)$ is invertible.

Then the critical point equation reduces to the finite-dimensional condition

$$\frac{\partial A}{\partial \eta}(\eta, \zeta(\eta, \omega), \omega) = 0, \quad (2)$$

equivalent to the condition that η be a critical point of the function on \mathcal{E} defined by $\eta \mapsto A(\eta, \zeta(\eta, \omega), \omega)$. This last equation is the *Liapunov-Schmidt reduction*. Solving it in a neighborhood of $\omega = 0$ is the topic of sections 4.4 and 5.

Remark. According to the Implicit Function theorem, the invertibility condition at η implies the existence of the solution $\zeta(\eta, \omega)$ for ω in some neighborhood $|\omega| < C_\eta$ of $\omega = 0$. The constant C_η can be taken independent of η , as can be seen by the equivariance of A , or, if one prefers, by the compactness of \mathcal{E} .

Before we continue, care is needed in understanding the above condition of invertibility of $\frac{\partial^2 A}{\partial \zeta^2}$. Note that we set $\omega = 0$ in writing this condition, so that we are dealing with the action functional A_0 . Write $H^s = H^s(S^1, \mathcal{X})$ for the Sobolev space of maps from the circle to \mathcal{X} which are square-integrable, together with their j th derivative, $j = 1, \dots, s$. Use the L^2 -pairing to convert differentials and quadratic forms to vectors and linear operators. Thus for $z \in H^2$, the one-form $dA_0(z)$ (a linear functional on H^1) becomes the L^2 -gradient $\nabla A_0(z) = -\ddot{z} + \nabla U(z)$ (lying in $L^2 = H^0$). The map $z \mapsto \nabla A_0(z)$ is a nonlinear map from H^2 to $L^2 = H^0$. And the Hessian of the action becomes

$$d^2 A_0(z)(w_1, w_2) = \langle L(z)w_1, w_2 \rangle,$$

where

$$L(z)w = -\ddot{w} + \nabla^2 U(z)w$$

and where we require that $w \in H^2$. (The Hessian $d^2A_0(\eta)$ is defined for $w_1, w_2 \in H^1$. However, in order for $L(\eta)w_1$ to be defined we need $w_1 \in H^2$.) Freeze z and write $L = L(z)$ for notational simplicity. L is the linearization of $\nabla A_0(z)$ at $z = (\eta, 0)$. L is formally self-adjoint and elliptic, and maps H^2 to L^2 . G -invariance of the action implies that $T_\eta \mathcal{E} \subset \ker(L)$. Self-adjointness of L implies that $\text{im}(L) \subset T_\eta \mathcal{E}^\perp$. The Poincaré non-degeneracy condition is precisely that $\ker(L) = T_\eta \mathcal{E}$ which is 4 dimensional. The index of L is zero by self-adjointness, or, alternatively, because L is a compact perturbation of the operator $-(d/dt)^2$ which has index 0. It follows that $\text{im}(L) = T_\eta \mathcal{E}^\perp \cap L^2$, the “ \perp ” being relative to L^2 . Write L_ζ for the restriction of L to $T_\eta \mathcal{E}^\perp$. L_ζ is what we mean by $\frac{\partial^2 A}{\partial \zeta^2}$ in the paragraph above when we were invoking the implicit function theorem. Then

$$L_\zeta : T_\eta \mathcal{E}^\perp \cap H^2 \rightarrow T_\eta \mathcal{E}^\perp \subset L^2.$$

is one-to-one and onto – an isomorphism. We have proved

Lemma 2 *The Poincaré non-degeneracy of the Eight η is equivalent to the invertibility of $\frac{\partial^2 A}{\partial \zeta^2}(\eta, 0, 0)$.*

Summary. We have established that the Poincaré non-degeneracy of the Eight implies the existence of an ω -dependent section $\zeta : \mathcal{T} \times U \subset \mathcal{T} \times \mathbb{R}^3 \rightarrow \mathcal{E}$, written $(\eta, \omega) \rightarrow \zeta(\eta, \omega)$ of the normal bundle $\mathcal{T} \rightarrow \mathcal{E}$ such that the second equation of (1) holds. The group G acts on $\mathcal{T}, \mathbb{R}^3$ and \mathcal{E} . The G -invariance of the action A_ω implies that this ω -dependent section map is G -equivariant:

$$\zeta(g\eta, g\omega) = g\zeta(\eta, \omega).$$

Remark. The invertibility of $\frac{\partial^2 A}{\partial \zeta^2}(\eta, 0, 0)$ is nothing but the *Bott non-degeneracy* of the critical manifold \mathcal{E} (see [B]). Hence the two notions of non-degeneracy coincide.

The next two sections are devoted to proving the Poincaré non-degeneracy of the Eight.

4.2. The Jordan structure of the monodromy operator (1)

Recall the linearization operator L of section 4.1:

$$(Lw)(t) = -\ddot{w} + \nabla^2 U(z(t))w(t),$$

where $z(t)$ denotes the standard Eight. Here, we do not insist that w need be periodic in t . The equation $Lw = 0$ is a system of six linear differential equation of second order; six because we fix the center of mass to be zero and are considering spatial solutions. The solution space \mathcal{S} to $Lw = 0$ is a 12 dimensional linear space which identifies with the three-body phase space $\mathbb{R}^{12} = \mathbb{R}^6 \oplus \mathbb{R}^6$ by associating to a solution w its initial conditions $w(0), \dot{w}(0)$. Write $\Phi_t : \mathbb{R}^{12} \rightarrow \mathbb{R}^{12}$ for the map taking $(w(0), \dot{w}(0))$ to $(w(t), \dot{w}(t))$. This map is the linearization of the Newton flow ϕ_t . The monodromy operator is

$$M = \Phi_T,$$

where T is the Eight’s period. Now, return to the L of section 4.1, whose domain consisted of vector-valued periodic functions: $w(t+T) = w(t)$. A function $w \in \mathcal{S}$ is in $\ker(L)$ if and only if $w(T) = w(0)$ and $\dot{w}(T) = \dot{w}(0)$, i.e. if and only if $MZ = Z$ where $Z = (w(0), \dot{w}(0))$. That is to say:

$$\ker(L) = \{w \in \mathcal{S} : Mw = w\}. \quad (*)$$

It follows that the Jordan block structure of the symplectic matrix M is the key to non-degeneracy.

For each first integral F of the flow ϕ_t , the value $X = X_F(a)$ of its Hamiltonian vector field X_F at a point a of the Eight is an eigenvector of M with eigenvalue 1 (i.e. $MX = X$). In the same way, $\ell = dF(a)$ is an eigencovector of M with eigenvalue 1 (i.e. $\ell \circ M = \ell$). In terms of matrix representation, X corresponds to a column vector while ℓ corresponds to a row vector (or left eigenvector) (see for instance [Me, p. 80, lemma 6.5.1]). Hence, to the four integrals of energy H , and angular momenta J_1, J_2, J_3 are associated four eigenvectors with eigenvalue 1. The following lemma and the fact that the angular momentum of the Eight is equal to 0 imply that the multiplicity of the eigenvalue 1 actually is at least 8.

Lemma 3 *Let $M : V \rightarrow V$ be a linear operator of a complex vector space of finite dimension n . Let X_1, \dots, X_p (resp. ℓ_1, \dots, ℓ_q) be p independent eigenvectors (resp. q independent eigencovectors) with eigenvalue 1. If moreover $l_i(X_j) = 0$ for all pairs (i, j) , $1 \leq i \leq q$, $1 \leq j \leq p$, the multiplicity of 1 as an eigenvalue of M is at least $p + q$.*

Proof. In a basis $\{e_1, \dots, e_n\}$ of V such that

- 1) $e_i = X_i$ for $1 \leq i \leq p$,
- 2) e_1, \dots, e_{n-q} generate the kernel of $(\ell_1, \dots, \ell_q) : \mathbb{R}^n \rightarrow \mathbb{R}^q$,
- 3) $\ell_i(e_{n-q+j}) = \delta_{ij}$ for all $1 \leq i, j \leq q$,

the matrix of M has the following form (the diagonal blocks are respectively of size $p, n - p - q, q$):

$$\begin{pmatrix} \text{Id} & * & * \\ 0 & * & * \\ 0 & 0 & \text{Id} \end{pmatrix},$$

which proves the lemma.

Now, two complex conjugate pairs $\exp(\pm i2\pi\lambda_1)$, $\exp(\pm i2\pi\lambda_2)$ lying on the unit circle were already numerically computed by Simó in his study of the stability of the Eight after reduction of the first integrals [S]. (Simó studies the planar problem but when the angular momentum is zero the motion is necessarily planar.) The angles λ_j are approximately

$$0.298\,092\,529\,004 \quad \text{and} \quad 0.008\,422\,724\,708.$$

From this fact and lemma 3 (with $p = q = 4$), we deduce

Lemma 4 *The spectrum of the monodromy operator M consists of the real number 1 with multiplicity 8 and the two simple complex conjugate pairs $e^{\pm i2\pi\lambda_1}$, $e^{\pm i2\pi\lambda_2}$, each with multiplicity 1. Moreover, the Jordan blocks corresponding to the eigenvalue 1 are at most 2×2 .*

Proof. As the angular momentum of the Eight is equal to 0, the four first integrals H, J_1, J_2, J_3 commute and hence the eigenvectors and covectors deduced from them satisfy the hypotheses of lemma 3. It remains only to prove the assertion on the size of the Jordan blocks of M . But the fact that the remaining eigenvalues are all different from 1 implies that we can find a new basis of V which converts M into a matrix of the form

$$\begin{pmatrix} \text{Id} & 0 & * \\ 0 & * & 0 \\ 0 & 0 & \text{Id} \end{pmatrix}.$$

It is now clear that the restriction $\text{Id} + N$ of M to the 8-dimensional generalized eigenspace associated to the eigenvalue 1 is block triangular and hence such that $N^2 = 0$, which proves the assertion.

4.3. The Jordan structure of the monodromy operator (2)

In this section, *which is not necessary for the sequel*, we elaborate on the jordanisation of M , paying attention successively to the symplectic structure, the “energy-scaling” block and the “horizontal-vertical” decomposition.

4.3.1. In our symplectic setting, we can give a simple interpretation of the non-diagonal terms in the Jordan blocks associated to the eigenvalue 1 of the monodromy operator M .

The vector space V is of dimension 12 and it splits as the sum $V = U \oplus E$ of two invariant symplectic subspaces, where the U is the 8-dimensional generalized eigenspace associated to the eigenvalue 1. We shall still call $M : U \rightarrow U$ the restriction to U of the monodromy map. The subspace L of U generated by the four eigenvectors $X_H(a), X_{J_1}(a), X_{J_2}(a), X_{J_3}(a)$ is Lagrangian, precisely because the angular momentum of the Eight equals 0. The restriction of M to L is the Identity, hence the following lemma applies:

Lemma 5 *Let $M : U \rightarrow U$ be a symplectic linear mapping from the symplectic vector space (U, ω) of dimension $2n$ to itself. Let $L \subset U$ be a Lagrangian subspace on which M induces the Identity. Then*

- 1) *the sole eigenvalue of M is 1;*
- 2) *the formula*

$$s(u', u'') = \omega(Mu' - u', u'')$$

defines a symmetric bilinear form on the quotient space U/L . This bilinear form is non-degenerate if and only if the Jordan form of M is made of n non-trivial 2×2 blocks. More precisely, the signature of s gives the number of blocks with positive (resp. negative) non-diagonal entry.

Proof. The fact that s is a well defined symmetric bilinear form is checked directly. It also follows from the computations below with coordinates. If one choses to identify U/L with a Lagrangian supplementary L' of L in U , this form becomes the *symplectic angular form* of the pair $(L', M(L'))$ which appears at the beginning of [LMS]. It is non-degenerate if and only if L' and $M(L')$ are transverse. (This transversality is independent of the choice of L' because M induces the Identity on L .) If one chooses symplectic coordinates (p, q) such that the p 's generate L and the q 's generate L' , the matrix of M will be of the form

$$\begin{pmatrix} \text{Id} & S \\ 0 & \text{Id} \end{pmatrix},$$

where the symmetric matrix S represents s . A linear symplectic change of variables preserving L is of the form

$$\begin{pmatrix} A & A\Sigma \\ 0 & (A^{-1})^t \end{pmatrix},$$

with Σ symmetric. This change transforms M into

$$\begin{pmatrix} \text{Id} & A^{-1}S(A^{-1})^t \\ 0 & \text{Id} \end{pmatrix}.$$

This allows us to transform S into $\text{diag}(s_1, \dots, s_n)$ with $s_i = 0, 1$ or -1 . Reordering the coordinates by conjugate pairs, this transforms the matrix of M into

$$\text{diag} \left[\dots \begin{pmatrix} 1 & s_i \\ 0 & 1 \end{pmatrix} \dots \right].$$

4.3.2. It seems that the numerical computations of the next paragraph are necessary to prove the non-degeneracy of s . Nevertheless, we know from general principles of at least one non-trivial 2×2 Jordan block, the "energy-scaling" block given by the following lemma:

Lemma 6 *Let $x(t)$ be a periodic solution of period T of the n -body problem in an Euclidean vector space E . Let $a = (x(0), \dot{x}(0)) = (x(T), \dot{x}(T))$ be a point on the corresponding integral curve of Newton flow $\varphi_t(x(0), \dot{x}(0)) = (x(t), \dot{x}(t))$ and $M = d\varphi_T(a)$ be the monodromy operator at a . If H is the corresponding Hamiltonian, the vectors $e_1 = X_H = (\dot{x}(0), \ddot{x}(0))$ and $e_2 = (-\frac{2}{3}x(0), \frac{1}{3}\dot{x}(0))$ generate an invariant plane on which M induces the non trivial Jordan block*

$$\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}.$$

Proof. We take the one parameter family of initial conditions

$$a_\lambda = (\lambda^{-\frac{1}{3}}x(0), \lambda^{\frac{1}{3}}\dot{x}(0)).$$

From the homothety symmetry described in the first paragraph of this article, we have $\varphi_t(a_\lambda) = (x_\lambda(t), \dot{x}_\lambda(t)) = (\lambda^{-\frac{2}{3}}x(\lambda t), \lambda^{\frac{1}{3}}\dot{x}(\lambda t))$. Differentiating with respect to λ at $\lambda = 1$ yields :

$$d\varphi_T(x(0), \dot{x}(0)) \left(-\frac{2}{3}x(0), \frac{1}{3}\dot{x}(0) \right) = \left(-\frac{2}{3}x(T) + T\dot{x}(T), \frac{1}{3}\dot{x}(T) + T\ddot{x}(T) \right).$$

If we set

$$e_1 = X_H = (\dot{x}(0), \ddot{x}(0)), \quad e_2 = \left(-\frac{2}{3}x(0), \frac{1}{3}\dot{x}(0) \right),$$

we get that e_1 and e_2 generate a M -invariant plane on which M is the said Jordan block.

4.3.3. Finally, let us call *horizontal* the plane containing the Eight, or more generally a periodic solution of the planar n -body problem, viewed within in \mathbb{R}^3 and *vertical* the orthogonal direction. It follows from Pythagoras' theorem that any vertical infinitesimal deformation does not change the mutual distances to first order. An easy consequence is the splitting of the tangent space V at a point a of the integral curve, isomorphic to \mathbb{R}^{12} , into the sum $V_h \oplus V_v$ of two subspaces, called horizontal and vertical, and invariant under the monodromy operator. In the case of the Eight, the horizontal space V_h is the direct sum of the eigenspaces associated to the elliptic eigenvalues (total dimension 4), the plane of the energy-scaling Jordan block, and the plane of the Jordan block with eigenvalue 1 and eigenvector X_{J_3} generated by the vertical component of the angular momentum. The vertical space V_v is the direct sum of the planes of the Jordan blocks with eigenvalue 1 and eigenvectors X_{J_1} and X_{J_2} .

4.4. Checking non-triviality of the Jordan blocks numerically

We have numerically checked the next lemma :

Lemma 7 *The monodromy operator M has four 2-dimensional Jordan blocks associated to the eigenvalue 1.*

The numerical verification consists of the steps below. The property of having non-trivial Jordan blocks is not open, hence computing the Jordan normal form of a numerical approximation of the monodromy operator M may lead to ludicrous results if no special care is taken.

- (i) Compute the monodromy operator M by integrating the linearized Newton equations along the Eight. We used the Odex algorithm [HNW] in extended precision, which was implemented in the C programming language by M. Gastineau. We computed the monodromy operator with a precision of 10^{-14} and found that it has the following approximate matrix:

$$\begin{pmatrix} -3.83 & -12.0 & -2.91 & 11.0 & 0.0 & 0.0 & -8.18 & -3.89 & -0.581 & 5.82 & 0.0 & 0.0 \\ 11.7 & 2.18 & 13.6 & -11.9 & 0.0 & 0.0 & -6.67 & 6.01 & -7.30 & -28.3 & 0.0 & 0.0 \\ 16.4 & 16.7 & 14.3 & -23.2 & 0.0 & 0.0 & 2.33 & 11.5 & -4.28 & -27.2 & 0.0 & 0.0 \\ 13.6 & 5.0 & 14.1 & -14.9 & 0.0 & 0.0 & -6.34 & 7.92 & -6.79 & -29.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -1.44 & 1.29 & 0.0 & 0.0 & 0.0 & 0.0 & 1.22 & 2.57 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.91 & 3.44 & 0.0 & 0.0 & 0.0 & 0.0 & -0.955 & 4.87 \\ -6.01 & 1.28 & -7.92 & 5.05 & 0.0 & 0.0 & 5.18 & -2.61 & 4.64 & 16.5 & 0.0 & 0.0 \\ -5.02 & 11.2 & -7.29 & -3.49 & 0.0 & 0.0 & 13.2 & -0.829 & 6.29 & 15.3 & 0.0 & 0.0 \\ 28.3 & 10.7 & 29.2 & -33.3 & 0.0 & 0.0 & -13.0 & 16.5 & -13.0 & -60.5 & 0.0 & 0.0 \\ -6.33 & 0.222 & -7.35 & 6.22 & 0.0 & 0.0 & 5.33 & -3.27 & 4.06 & 16.3 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -4.87 & 2.57 & 0.0 & 0.0 & 0.0 & 0.0 & 3.44 & 5.14 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.955 & -1.22 & 0.0 & 0.0 & 0.0 & 0.0 & 0.478 & -1.44 \end{pmatrix}$$

More digits can be given on demand. The basis we used is the basis associated with symplectic ‘heliocentric’ coordinates

$$(x_1, X_1, y_1, Y_1, z_1, Z_1, x_2, X_2, y_2, Y_2, z_2, Z_2),$$

the Euler configuration of the three bodies, masses $m = 1$, and period $T = 2\pi$. Our heliocentric coordinates are defined in a Gallilean frame of reference where the center of mass is at the origin by $(x_j, y_j, z_j) = q_j - q_3$, $j = 1, 2$, where q_j , $j = 1, 2, 3$, is the position vector of the j -th body, and where (X_j, Y_j, Z_j) , $j = 1, 2$, is its linear momentum. Approximate initial conditions in these coordinates are

$$\begin{pmatrix} x_1 & X_1 \\ y_1 & Y_1 \\ z_1 & Z_1 \end{pmatrix} = \begin{pmatrix} -0.965631465375792 & -0.934516204162903 \\ -0.241991664955778 & 0.866687230530148 \\ 0 & 0 \end{pmatrix}.$$

- (ii) Compute the matrix of M in a basis chosen as in the proof of lemma 3. We used Maple to symbolically compute the eigenvectors associated to the 4 first integrals, and to numerically compute the eigenvectors associated to the 4 non-trivial simple eigenvalues. The approximate eigenvalues we find agree with the values λ_j of Simó given above.
- (iii) Make adequate linear combinations of the last 8 vectors of the latter basis, in order to compute a basis similar to that of the proof of lemma 4. Then compute the matrix of M in the new basis, and the matrix $\text{Id} + N$ of the operator induced by M on the generalized eigenspace of the eigenvalue 1.

(iv) Check that N has rank 4 (and, as already known, that $N^2 = 0$).

Corollary 8 *The real Jordan block structure of the monodromy operator M is the following :*

- associated to each complex conjugate pair of eigenvalues is an invariant real two-dimensional symplectic subspace E_j of \mathcal{S} , endowed with a complex structure, so that M acts on E_j by rotation by $2\pi\lambda_j$ radians;
- the generalized eigenspace for 1 is an eight-dimensional symplectic space which decomposes into four symplectic subspaces of dimension two corresponding to four isomorphic 2-dimensional Jordan blocks.

End of the proof of the non-degeneracy

It follows from the above lemma that the eigenspace for 1 for M is four-dimensional and has X_H and the X_{J_i} as a basis. These are the infinitesimal generators of the symmetry algebra G and consequently they form a basis for $T_z\mathcal{E}$. Referring to (*) at the beginning of section 4.2 we now see that indeed $T_z\mathcal{E} = \ker(L)$. This ends the proof of the non-degeneracy of the Eight.

4.5. Reduction to a nonlinear eigenvalue equation

Having checked the invertibility of $\frac{\partial^2 A}{\partial \zeta^2}(\eta, 0, 0)$, we remain with the problem of solving the reduced equation $\frac{\partial A}{\partial \eta}(\eta, \zeta(\eta, \omega), \omega)$. To compute the derivative of A with respect to η we use the invariance of the parameterized Lagrangian under the G -action. We must compute the derivative of the action with respect to an arbitrary variation $\Delta\eta \in T_\eta\mathcal{E}$. Now $T_\eta\mathcal{E}$ is the direct sum of the space infinitesimally generated by the spatial rotations $SO(3) \subset G$ and the line $span(\dot{\eta})$ infinitesimally generated by the time translations $SO(2) \subset G$. Because the action is invariant under time translations and because these act trivially on the parameter ω , the derivative of A with respect to the time translation direction is zero. It remains to compute the derivative of A with respect to a variation $\Delta\eta$ generated by an arbitrary one-parameter subgroup R_ϵ of $SO(3)$.

Let $R_\epsilon = \exp(\epsilon X)$ be such a subgroup. Write $\Delta\eta = \frac{d}{d\epsilon}[R_\epsilon\eta]_{\epsilon=0}$. Then

$$\frac{\partial A}{\partial \eta}(\eta, \zeta, \omega)\Delta\eta = \frac{d}{d\epsilon}[A(R_\epsilon\eta, \zeta, \omega)]_{\epsilon=0}$$

As $\frac{\partial A}{\partial \zeta}(\eta, \zeta(\eta, \omega), \omega) = 0$, this implies

$$\frac{\partial A}{\partial \eta}(\eta, \zeta(\eta, \omega), \omega)\Delta\eta = \frac{d}{d\epsilon}[A(R_\epsilon\eta, R_\epsilon\zeta(\eta, \omega), \omega)]_{\epsilon=0}.$$

The rotational invariance of A and the equivariance of the section $\zeta(\eta, \omega)$ imply that

$$A(R_\epsilon\eta, R_\epsilon\zeta, \omega) = A(\eta, \zeta, R_{-\epsilon}\omega).$$

Consequently

$$\frac{\partial A}{\partial \eta}(\eta, \zeta(\eta, \omega), \omega)\Delta\eta = \frac{d}{d\epsilon}[A(\eta, \zeta(\eta, \omega), R_{-\epsilon}\omega)]_{\epsilon=0}.$$

We can compute this last derivative explicitly from the formula for L_ω and the fact that $\frac{d}{d\epsilon} [R_{-\epsilon}\omega]_{\epsilon=0} = -X \wedge \omega$ where X and ω are to be viewed as vectors in \mathbb{R}^3 and the wedge is their cross product. Thus:

$$\begin{aligned} \frac{\partial A}{\partial \eta}(\eta, \zeta(\eta, \omega), \omega) \Delta \eta &= \langle -X \wedge \omega, \mathbb{I}(\eta, \zeta(\eta, \omega))\omega \rangle + \langle -X \wedge \omega, \mathbb{J}(\eta, \zeta(\eta, \omega)) \rangle \\ &= \langle \mathbb{I}(\eta, \zeta(\eta, \omega))\omega + \mathbb{J}(\eta, \zeta(\eta, \omega)), \omega \wedge X \rangle \\ &= \langle [\mathbb{I}(\eta, \zeta(\eta, \omega))\omega + \mathbb{J}(\eta, \zeta(\eta, \omega))] \wedge \omega, X \rangle. \end{aligned}$$

Equation (2) of section 4.1 for a critical point of A_ω becomes

$$[\mathbb{I}(\eta, \zeta(\eta, \omega))\omega + \mathbb{J}(\eta, \zeta(\eta, \omega))] \wedge \omega = 0.$$

This is a non-linear eigenvalue problem. Indeed

$$\omega \mapsto B(\eta, \omega) := \mathbb{I}(\eta, \zeta(\eta, \omega))\omega + \mathbb{J}(\eta, \zeta(\eta, \omega)).$$

is a map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ and the condition of criticality is that there is a scalar $\lambda = \lambda(\eta, \omega)$ such that

$$B(\eta, \omega) = \lambda\omega.$$

5. Solution by group theory

It follows from the integral formulae for \mathbb{I} and \mathbb{J} and from the equivariance of the implicit function section $\zeta(\eta, \omega)$ that

$$B(g\eta, g\omega) = gB(\eta, \omega)$$

where the action of g on ω and on $B(\eta, \omega)$ is thru the action of $g \in G$ on bivectors described in section 3.2 above. We will use this equivariance and the discrete symmetries of the Eight to solve our nonlinear eigenvalue equation for B .

Fix $\eta = x_8$, the standard Eight, lying in the xOy plane. Recall that Γ is its isotropy group, so that $gx_8 = x_8$ for $g \in \Gamma$. It follows from the equivariance of B that then

$$B(x_8, g\omega) = gB(x_8, \omega), \text{ for } g \in \Gamma.$$

Now suppose that some particular element $g \in \Gamma$ has the property that its fixed point set within the space of bivectors:

$$Fix(g) = \{\omega \in \Lambda^2\mathbb{R}^3 : g\omega = \omega\} \subset \Lambda^2\mathbb{R}^3$$

is one-dimensional. If we choose $\omega \in Fix(g)$ then $B(\eta, \omega) = B(\eta, g\omega) = gB(\eta, \omega)$ so that we also have that $B(\eta, \omega) \in Fix(g)$. By the one-dimensionality of $Fix(g)$ we have that $B(\eta, \omega) = \lambda\omega$. We have found a solution to our nonlinear eigenvalue equation! We do not claim, although we suspect, that all solutions are obtained this way (see the note at the end).

Our isotropy group Γ is equal to $\Gamma = D_6 \times Z_2$ where the first D_6 factor is the isotropy group of the planar Eight as used in its rediscovery [CM] and acts on the plane of the Eight. The second Z_2 factor is generated by the reflection S across the plane of the Eight, taken to be the xOy plane. The group Γ has the presentation

$$\Gamma = \{s, \sigma, r \mid s^6 = 1, \sigma^2 = 1, r^2 = 1, s\sigma = \sigma s^{-1}, sr = rs, \sigma r = r\sigma\}.$$

with generators acting on loops $\xi(t) = (\vec{\rho}_1(t), \vec{\rho}_2(t), \vec{\rho}_3(t))$ according to

$$\begin{aligned}(s \cdot \xi)(t) &= (\Sigma \vec{\rho}_3(t - T/6), \Sigma \vec{\rho}_1(t - T/6), \Sigma \vec{\rho}_2(t - T/6)) \\ (\sigma \cdot \xi)(t) &= (\Delta \vec{\rho}_1(-t), \Delta \vec{\rho}_3(-t), \Delta \vec{\rho}_2(-t)) \\ (r \cdot \xi)(t) &= (S \vec{\rho}_1(t), S \vec{\rho}_2(t), S \vec{\rho}_3(t)),\end{aligned}$$

where Σ denotes the symmetry with respect to the yOz plane, S is the symmetry with respect to the xOy plane, and Δ denotes the symmetry with respect to the Oz line. Here ‘‘symmetry’’ with respect to a plane Π means reflection through this plane while ‘‘symmetry’’ with respect to a line ℓ means rotation by 180 degrees about this line. The symmetry axes of the Eight are the x -axis, y -axis, and z -axis. We write e_1, e_2, e_3 for a basis aligned with these axes respectively.

Take $g = (\rho, \sigma, \tau) \in O(3) \times O(2) \times S_3 = G$. We observe the following facts which follow from section 2.2. If ρ is the symmetry about the plane Π and if $\sigma \in O(2)$ is orientation preserving then $Fix(g)$ is the line Π^\perp . If ρ is the symmetry about the line ℓ and if σ is orientation preserving then $Fix(g)$ is the line ℓ . On the other hand, if ρ is the symmetry about the plane Π and if $\sigma \in O(2)$ is orientation reversing then $Fix(g)$ is the plane Π , and if ρ is the symmetry about the line ℓ and if σ is orientation reversing then $Fix(g)$ is the plane ℓ^\perp . It follows from these considerations that

$$Fix(s) = Span(e_1), \quad Fix(rs) = Span(e_2), \quad Fix(r) = Span(e_3).$$

For a subgroup $\Gamma' \subset \Gamma$ we write

$$Fix(\Gamma') = \{\omega \in \Lambda^2 \mathbb{R}^3 : \gamma \omega = \omega \text{ for all } \gamma \in \Gamma'\} \subset \Lambda^2 \mathbb{R}^3.$$

Consider the following three subgroups of Γ :

Γ_1 generated by s, σ , isomorphic to D_6 ,

Γ_2 generated by rs, σ , isomorphic to D_6 ,

Γ_3 generated by $s^2, s\sigma, r$, isomorphic to $D_3 \times \mathbb{Z}_2$ and hence to D_6 .

Lemma 9 *We have $Fix(\Gamma_i) = Span(e_i)$, the i th symmetry axis of the Eight. Moreover Γ_i is maximal among all subgroups of Γ whose fixed point set is $Span(e_i)$.*

What remains of the proof of lemma 9 is left to the reader.

Lemma 10 *If the subgroup $\Gamma' \subset \Gamma$ has the property that $Fix(\Gamma')$ is one-dimensional then any $\omega \in Fix(\Gamma')$ yields a solution to our nonlinear eigenvalue equation $B(\eta, \omega) = \lambda(\eta, \omega)\omega$. Consequently such an ω yields a solution $\xi(\omega)(t)$ to Newton’s equation which is periodic with respect to the rotating frame defined by ω . Moreover this rotating solution has isotropy group in the rotating frame at least as big as Γ' .*

Proof of Lemma 10. At least one element $\gamma \in \Gamma'$ must have $Fix(\gamma) = Fix(\Gamma')$, a line. If ω spans this line then $B(\eta, \omega) = \lambda\omega$ as discussed in the beginning of this section. It follows that for such ω , sufficiently close to zero we obtain a solution $\xi(\omega) = x_8 + \zeta(x_8, \omega)$ as in the second paragraph of this section. (Recall the splitting of section 3.3 and the Liapunov-Schmidt reduction, section 4.1.) For any other $\gamma' \in \Gamma'$ we have $\gamma'\omega = \omega$ and so, by equivariance, $\gamma'\xi(\omega) = \xi(\omega)$.

Proof of theorem 1. Combine lemmas 2 and 10.

6. Commentary

Refer to each family of the theorem by its subgroup Γ_i . Following Marchal we will also refer to the family Γ_1 , discovered by Marchal, as the ‘ P_{12} ’ family.

- (i) It was explained in [Ma2] (see also [C1] and [C2]) how the P_{12} family could be obtained through minimization of the action in the *fixed* frame: one looks for a minimizer of the action among paths which start at $t = 0$ in a configuration which is symmetric with respect to the Oz axis with body 1 on the axis of symmetry and end at $t = T/12$ in a configuration which is symmetric with respect to the image P_u of the xOz plane by a rotation of angle u around the x axis with body 3 on P_u . Here $0 \leq u \leq \pi/6$ and is the rotation of the rotating frame during the interval of time. In the same way, the restriction to the interval $[0, T/12]$ of a member of the Γ_2 family starts being symmetric with respect to the Oz axis with body 1 on the axis of symmetry and ends being symmetric with respect to the image D_u of the Ox axis by a rotation of angle $-u$ around the Oy axis with body 3 on D_u (the plane zOx is oriented as written). But this restriction *does not* minimize the action among paths with this behaviour: indeed, it is enough to notice that, already for $u = 0$, the action of the Eight is larger than that of the retrograde Lagrange equilateral solution of the same period in the zOx plane with body 1 starting on the Oz axis. Finally, the restriction to the interval $[-T/12, +T/12]$ of a member of the Γ_3 family (necessarily lying in the xOy plane) starts in a configuration symmetric with respect to the image H_{-u} of the Ox axis by a rotation of angle $-u$ around Oz with body 2 on H_{-u} and ends in a configuration symmetric with respect to the image H_u of the Ox axis by a rotation of angle u around Oz with body 3 on H_u . Here also, comparison with the Lagrange solution in the xOy plane with the same symmetries and the same period ($u = 0$) shows that this restriction cannot be obtained by minimization. The peculiarity of the Γ_1 family is that, for $u = 0$, one needs to describe *twice* a Lagrange relative equilibrium of half the period in the yOz plane in order to satisfy the symmetry requirements; but the action of this last solution is larger than that of the Eight.
- (ii) The proof of the theorem allows for the existence of symmetry-breaking bifurcations in which the maximal subgroups $\Gamma_1, \Gamma_2, \Gamma_3$ of lemma 1 are replaced by their minimal subgroups $\gamma_1, \gamma_2, \gamma_3$, respectively generated by s, rs, r , and of orders 6, 6, 2. If unicity holds, which as we said at the beginning is very likely, such bifurcations do not lead to new less symmetric solutions
- (iii) Reversing the direction of traversing the Eight is equivalent to rotating it by π in the same plane. Consequently, the Eight does not provide any orientation of its plane, and so reversing the rotation of the frame does not change the bifurcating family Γ_i , up to rotation.
- (iv) In particular, the P_{12} family may be considered as joining the direct Lagrange solution to the retrograde one of the same period through choreographies in the rotating frame (from -2π to $+2\pi$, the middle value 0 corresponding to the Eight).
- (v) The existence of the third family was originally postulated on grounds of topology. The space of all Eights modulo rotation about a fixed axis in space (thought of as the axis of a rotating frame) forms a real projective plane P^2 . Any smooth function on P^2 has at least three critical points. The idea was that two of these were the known branches of Hénon and Marchal, and the third critical

point corresponded to an as-yet-to-be discovered third branch Γ_2 . We learned later of the existence of the paper [ACZE] (especially Remark 2.7 at the end of page 170) where it is shown that A_ω possesses at least as many critical points bifurcating from \mathcal{E} as the Liusternik-Schnirelman category of the quotient of \mathcal{E} by the remaining symmetries (rotation around the axis of ω and time translations). But it is precisely this quotient which is diffeomorphic to the projective plane, whoses category is 3. In a sense, we reprove this assertion but with more precision due to our study of the symmetry subgroups Γ_i .

- (vi) Another hint of the existence of the third family, rotating around Oy , is purely mechanical: the velocities along the two branches of the Eight at its crossing point have the same projection along the Oy axis and opposite ones along the Ox axis. This explains why a rotation around the Ox axis keeps together the two branches while, on the contrary, a rotation around the Oy axis separates the two branches.
- (vii) Following the idea of lemma 6, we can use the existence of the three Γ_i families to exhibit a symplectic Jordan basis for the monodromy matrix M . In order to do this, rescale the solutions of each family using the homothety symmetry so that the new solutions all have the same energy (rather than the same period) as the original Eight. Name these rescaled curves of solutions $\tilde{\Gamma}_i(s)$ with curve parameter s taken so that $s = 0$ represents the Eight x_8 . Add a new branch $\tilde{\Gamma}_0$ which is simply the x_8 rescaled using homotheties. Apply the 4-dimensional symmetry group G to these curves to obtain four 5-dimensional families \mathcal{F}_i , $i = 1, 2, 3, 4$ of solutions whose common intersection is the space \mathcal{E} of Eights. The identification of a solution $x(t)$ with the point $(x(0), \dot{x}(0))$ of the phase space \mathbb{R}^{12} transforms \mathcal{E} into the 4-dimensional submanifold $\hat{\mathcal{E}}$ of \mathbb{R}^{12} , the curves $\tilde{\Gamma}_i$ into four immersed curves $\hat{\Gamma}_i$ passing through $a = (x_8(0), \dot{x}_8(0))$, and the \mathcal{F}_i into four 5-dimensional immersed submanifolds $\hat{\mathcal{F}}_i$ whose common intersection is $\hat{\mathcal{E}}$. Moreover, if $i \neq j$ then J_i is constant on $\hat{\mathcal{F}}_j$, where we set $J_0 = H$ for convenience. The four vectors $X_i = X_{J_i}(a)$, $i = 0, 1, 2, 3$ form a basis for the tangent space $T_a \hat{\mathcal{E}}$. The vectors $Y_i = d\tilde{\Gamma}_i/ds$ at $s = 0$ are tangent to $\hat{\mathcal{F}}_i$ and transverse to $\hat{\mathcal{E}}$ at a . If ω is the symplectic form, we claim that $\omega(X_i, Y_j)$ is 0 for $i \neq j$ and that for $i = j$ we can normalize Y_i so that $\omega(X_i, Y_i) = 1$. The first assertion follows from $\omega(X_i, Y_j) = dJ_i(a)Y_j$. The only thing which must be checked (and which is true numerically) is that the i th component of the angular momentum indeed does vary along $\hat{\mathcal{F}}_i$ for $i = 1, 2, 3$. Finally, there exist real numbers ρ_i^j such that if $Y'_i = Y_i + \sum_j \rho_i^j X_j$ is the subspace L' generated by the Y'_i is Lagrangian in the 8-dimensional generalized eigenspace U of 1 and hence the basis formed by the X_i and the Y'_i is a symplectic basis of U which jordanizes the restriction of M to U . Note that Y'_i is still tangent to \mathcal{F}_i for $i = 0, 1, 2, 3$. That the basis is symplectic amounts to showing that $\omega(Y'_i, Y'_j) = 0$ and this is equivalent to the system of 6 equations and 12 unknown $\rho_i^j - \rho_j^i = -\omega(Y_i, Y_j) = 0$ whose resolution is obvious. The assertion on the Jordan form in the basis X_i, Y'_i is a translation of the behaviour of the families of relative periodic solutions. Changing to the basis X_i, Y'_i does not change the matrix.

7. Numerical pictures of the three families

Numerical pictures of the Γ_3 -family were first obtained by Hénon [H2]. Figures 4 and 5 of [Nau], obtained independently of Hénon and Marchal by a steepest descent with Fourier series, respectively show a member of the Γ_3 family between our figures 12 and 13, and a member of the Γ_1 family as in our figure 3 (a typo in the caption has replaced Ox by Oz). Figures of the Γ_2 family appear for the first time. The figures below are obtained in the following way :

For each of the three Γ_i families, the symmetry conditions at the beginning (resp. at the end) of a fundamental interval of the action of the group on the time axis define a 6-dimensional subspace of the 12-dimensional phase space. Indeed, as the center of mass is fixed at the origin, knowledge of the position and velocity of anyone of the bodies not belonging to the axis or plane of symmetry is sufficient to define uniquely the position and velocity of the two others. This implies that, generically, the intersection of the image under the flow of the admissible initial conditions with the admissible final conditions will consist of isolated points, corresponding to isolated solutions. Starting from the Eight, one gets without ambiguity the Γ_i family we are seeking. The table below shows the stabilizers of the endpoints of a fundamental time interval for each Γ_i action.

	Γ_1	Γ_2	Γ_3
Fundamental interval $[t_0, t_1]$	$[0, T/12]$	$[0, T/12]$	$[-T/12, T/12]$
Stabilizer of $\{t_0\}$	σ	σ	$\{s^5\sigma, r\}$
Stabilizer of $\{t_1\}$	$s\sigma$	$rs\sigma$	$\{s\sigma, r\}$.

We have numerically integrated Newton's equations with a Runge-Kutta algorithm of order 7 due to Dormand and Prince, with stepsize control [HNW]. The routine was implemented by F. Joutel and M. Gastineau. We have numerically solved the isolated solutions which were alluded to above, with a finite difference version of the so-called hybrid algorithm which is implemented in the Gnu Scientific Library. The plots have been drawn with the Gnuplot program.

The numerical data suggest that only the P_{12} family, corresponding to the Γ_1 symmetry ends with a Lagrange equilateral relative equilibrium solution. The explanation behind this fact can be found by studying the bifurcations of relative periodic solutions from the Lagrange relative equilibrium. This explanation is the object of the subsequent paper [CFM].

Final remark about the linear stability of the Γ_i families. We have numerically computed the bifurcations of the spectrum of the derivative of the return map φ_T after one period ; the derivative was computed at the point on the orbit corresponding respectively to $t = 0$ for Γ_1 and Γ_2 and $t = -T/12$ for Γ_3 . By (linear) stability we mean of course the only one which is possible, i.e. the stability after reduction of the angular momentum and energy integrals. Stability or instability is at first dictated by the way (complex or real) the eigenvalue 1 bifurcates, its multiplicity falling from 8 in the fixed frame to 4 in the rotating frame. The Γ_1 ($= P_{12}$) family becomes unstable as soon as the rotation number ω of the frame becomes different from 0. The instability appears through the bifurcation of a pair of real eigenvalues from the multiple eigenvalue 1 of the Eight. Another bifurcation, where two eigenvalues on the unit circle coalesce at 1 and become real, occurs when ω is around 0.9404. It corresponds to the branching off from P_{12} of a family of minimax solutions found by

Vivina Barutello in [Ba]. The Γ_2 family stays stable till ω is approximately 0.092, where a quadruple of eigenvalues pops out from the unit circle. Finally, the planar Γ_3 family stays stable until ω is at least 0.6, a value for which the curve has already developed a third lobe (see figure 13).

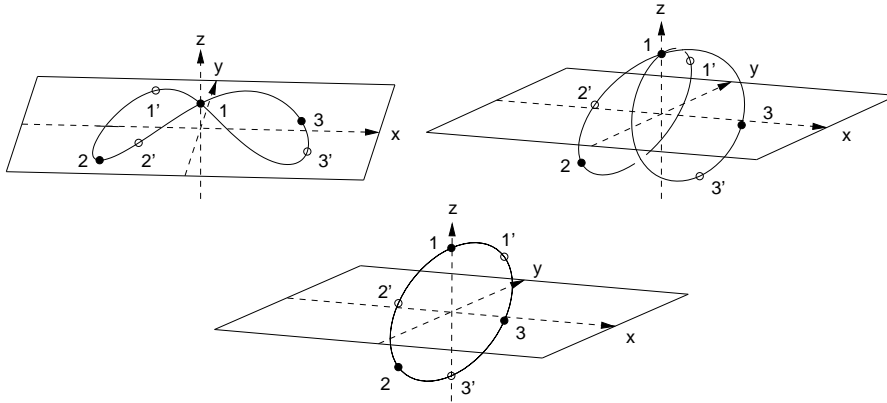


Figure 2. Two Γ_1 -symmetric solutions in a frame rotating around the x -axis ($\omega_x = 0.8$ and 0.97 respectively), followed by the Lagrange solution in a frame rotating around the x -axis ($\omega_x = 1$). The bodies are represented at time $t = 0$ by filled circles and at time $t = T/12$ by hollow circles.

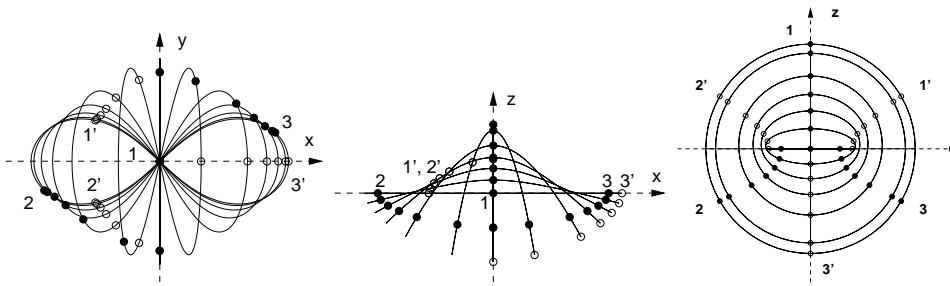


Figure 3. The Γ_1 -family, projected respectively onto the xy , xz and yz planes ($\omega_x = 0, 0.2, 0.4, 0.6, 0.8, 0.97, 1$).

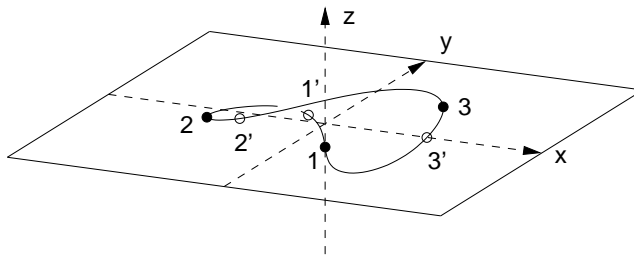


Figure 4. A Γ_2 -symmetric solution in a frame rotating around the y -axis with angular velocity $\omega_y = 0.8$. The bodies are represented at time $t = 0$ by filled circles and at time $t = T/12$ by hollow circles.

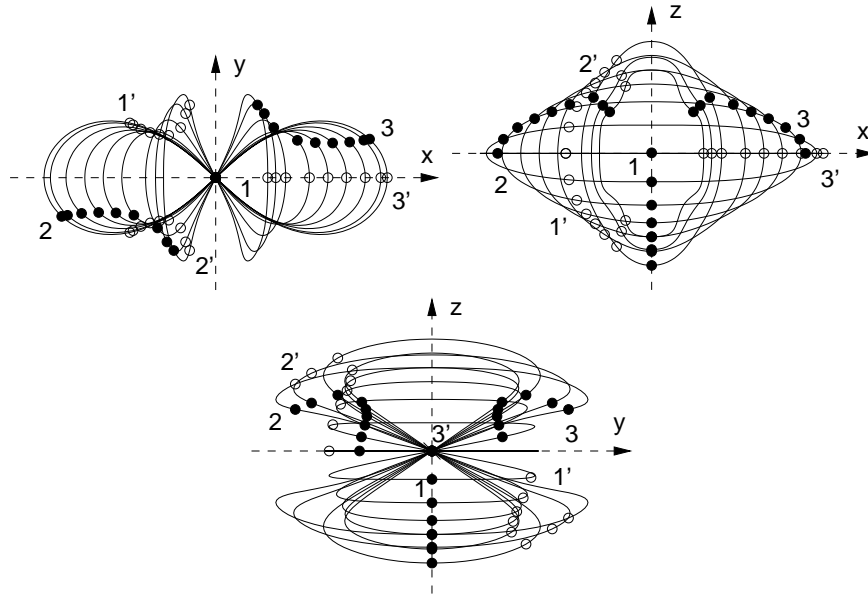


Figure 5. The Γ_2 -family, projected respectively onto the xy , xz and yz planes ($\omega_y = 0.2k$, $k = 0, \dots, 8$).

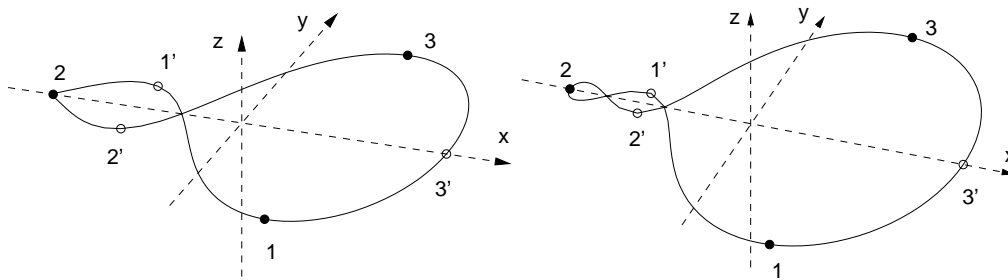


Figure 6. Two Γ_3 -symmetric solutions in a frame rotating around the z -axis with respective angular velocities $\omega_z = 0.4$ and 0.6 . The bodies are represented at time $t = -T/12$ by filled circles and at time $t = T/12$ by hollow circles.

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References

- [ACZE] Ambrosetti A, Coti Zelati V and Ekeland I 1987 Symmetry Breaking in Hamiltonian Systems *J. Differential Equations* **67** 165–84
- [Ba] Barutello V 2004 On the n -body problem *Thesis* (University of Milano-Bicocca) chapter 2
- [B] Bott R 1982 Lectures on Morse Theory, Old and New *Bulletin of the AMS* **7:2**, 331–58
- [C1] Chenciner A 2002 Action minimizing solutions of the n -body problem: from homology to symmetry *Proc. ICM Beijing* vol III, 279–94
- [C2] Chenciner A 2003 Some facts and more questions about the Eight, *Topological Methods, Variational Methods*, eds H Brezis *et al* (World Scientific) 77–88
- [CFM] Chenciner A, Féjóz J and Montgomery R 2005 Rotating Eights II : from Lagrange to the Eight *Preprint*
- [CM] Chenciner A and Montgomery R 2000 A remarkable periodic solution of the three body problem in the case of equal masses *Annals of Math.* **152** 881–901
- [CH] Chillingworth D R J 1991 The Signorini perturbation scheme in abstract setting *Proc. of the Royal Society of Edinburgh* **119A** 373–95
- [Fe] Ferrario D L and Terracini S 2004 On the existence of collisionless equivariant minimizers for the classical n -body problem *Invent. Math.* **155:2** 305–62
- [HNW] Hairer E Nørsett S and Wanner G 1993 Solving ordinary differential equations *Springer*
- [H1] Hénon M 1974 Families of periodic orbits in the three-body problem *Celestial Mech. Dynam. Astronom.* **10** 375–88
- [H2] Hénon M 2000 Private communication of computer pictures similar to those of figure 6
- [K] Kapela T and Zgliczynski P 2003 An existence of simple choreographies for N-body problem – a computer assisted proof *Nonlinearity* **16** 1899–918
- [LMS] Lochak P, Marco J-P and Sauzin D 2003 On the Splitting of Invariant Manifolds in Multidimensional Near-Integrable Hamiltonian Systems, *Memoirs of the AMS* **775**
- [Ma1] Marchal C 1990 *The 3-body problem*, Elsevier, paragraph 10.8.2
- [Ma2] Marchal C 2000 The family P_{12} of the three-body problem. The simplest family of periodic orbits with twelve symmetries per period *Celestial Mech. Dynam. Astronom.* **78** 279–98
- [Me] Meyer K 1999 Periodic Solutions of the N-Body Problem *LNM* (Springer) **1719**
- [MeLa] Meyer K and Laub A J 1974 Canonical forms for symplectic and hamiltonian matrices *Celestial Mech. Dynam. Astronom.* **9**, 213–38
- [Mo] Moore C 1993 Braids in Classical Gravity *Phys. Rev. Lett.* **70** 3675–9
- [Nau] Nauenberg M 2001 Periodic orbits for three particles with finite angular momentum *Phys. Lett. A* **292** 93–9
- [S] Simó C 2002 Dynamical properties of the figure Eight solution of the three-body problem *Contemporary Mathematics* **292** 209–28