How much does the rigid body rotate? A Berry's phase from the 18th century

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A formula, apparently new, is derived for the spatial rotation of a free rigid body during one period T of its body angular momentum vector. This formula has the structure of recent phase formulas of Berry [Proc. R. Soc. London Ser. A **392**, 45–57 (1984); Sci. Am. **259** (6), 46–55 (1988)] and Hannay [J. Phys. A **18**, 221–230 (1985)]; namely, it consists of a geometric and a dynamic part. It is derived by applying Stokes' theorem to a certain closed curve in phase space.

I. INTRODUCTION

A. Phase formula

The angular momentum vector of a free rigid body is constant in an inertial frame. When viewed from a body-fixed frame, the motion of this vector is periodic in time for typical initial conditions. In one such period, the body, as viewed from the inertial frame, must rotate about its angular momentum vector. What is the angle, $\Delta\theta$, of this rotation?

The purpose of this note is to derive the answer:

$$\Delta\theta = 2ET/J - \Omega, \tag{1}$$

where E is the kinetic energy of the initial condition, J is the length of the angular momentum vector, T is the period of its motion as viewed in the body frame, and Ω is the solid angle swept out by the angular momentum vector, when viewed from the body frame.

As usual, the Ω in this formula is a signed solid angle. It is positive or negative according to the usual right-hand rule. This means that if the motion of this body angular momentum vector is counterclockwise relative to a vector in the interior of the solid angle, then we call the solid angle positive, and if the motion is clockwise then we say that the solid angle is negative. (Mod 2π, this solid angle is independent of what we call "interior" and "exterior.")

B. Context and motivation

This formula is an example of a general class of phenomena recently popularized by Michael Berry[1,2] and John Hannay[3] as "geometric phases" and by others[4,5] as "Berry's phases." Suppose that some set of variables (the body angular momentum for us) undergoes a closed circuit, and that by virtue of the dynamical equations, that this circuit induces an angular variable θ (for us the angle of rotation) to suffer a change $\Delta\theta$. Then this change can often be expressed in the form $\Delta\theta =$ dynamic phase + geometric phase.

The dynamic phase ($2ET/J$ for us) can usually be guessed at, either by dimensional analysis, or by analyzing the situation where the closed circuit is a single point (which for our case occurs when the angular momentum is parallel to one of the body's principal axes of inertia). The geometric phase ($-\Omega$ in our case) has its name because it depends only on the geometry of the closed circuit. In particular, it is independent of the speed at which the curve is traversed. The geometric phase can be expressed as the Wilson loop integral of some gauge potential or, in mathematical terms, as the holonomy of some connection. In our case this connection is $1/J$ times the canonical one-form $p\,dq$ on the phase space of the rigid body. In Berry's case, the connection is also the canonical one-form, but on the Hilbert space of the quantum system. In both cases the formula for $\Delta\theta$ is obtained by integrating the canonical one-form around a loop and then applying Stokes' theorem to relate this to a surface integral.

We would like to emphasize that our formula for $\Delta\theta$ is exact. The reader may be aware that Berry[1] invoked an adiabatic approximation in order to obtain his original phase formula. This approximation is used only to insure that the circuit of quantum states is approximately closed. Roughly speaking, the formula is as exact as the circuit is closed. If, as in our case, one knows *a priori* that the circuit is closed then the phase formula is exact. Aharanov and Anandan[6] were the first to point out that Berry's phase formula is exact if the circuit of states exactly closes.

For more information concerning geometric phases in general, we recommend Refs. 4, 5, and 7.

C. Other derivations and expressions for the rigid-body phase·

Equation (1) first appeared in the review article[8] on classical geometric phases. The derivation there is based on the theory of connections (gauge fields) on principal bundles.

There is now another derivation of our Eq. (1) available, which is due to Mark Levi.[9] His derivation is based on Poinsot's rolling description of the motion of a free rigid body, together with some facts concerning the geometry of curves on the sphere.

Equation (1) really depends only on J, E, and the principal moments of inertia. This is because the period T, and the solid angle Ω, are functions of these parameters alone. Landau and Lifshitz[10] give such a formula for T [Eq. (37.12)]. It involves complete elliptic integrals. Such a formula for Ω can also be derived from the results in Sec. 37 of their text.

An alternative to our formula for $\Delta\theta$ can then be obtained by combining their equations (37.17) and (37.20). The result is that $\Delta\theta = \phi_2(T) = cT$, where the number c depends implicitly on J, E, and the moments of inertia through a set of transcendental equations involving elliptic theta functions. [It would be interesting to know if the equality between this formula and our Eq. (1) is a new identity for theta functions.]

D. Setup

The motion of a rigid body is described by a time-dependent 3×3 rotation matrix $g = g(t)$. To do this we fix an inertial frame with origin at the body's center of mass and fix a reference configuration of the body. If $\mathbf{X}$ is the position of a point on the reference body then

$$\mathbf{x} = g\cdot\mathbf{X} \tag{2}$$

is its position in the inertial frame. In particular, $g = I$, the 3×3 identity matrix, corresponds to the reference configuration.

A rotation matrix is any 3×3 matrix g that satisfies $gg^t = I$ (g is orthogonal) and $\det(g) = 1$ (no reflections). The superscript t denotes transpose. These properties of g are direct consequences of the fact that the distances between all points of the body must remain constant, and the fact that the motion is continuous. The set of rotation matrices can be coordinatized by the Euler angles if need be.

If the only forces acting on the body are the ones holding it together (the forces of constraint), then it is called a free rigid body. In this case its total angular momentum vector

$$\mathbf{M} = \sum \mathbf{x}_a \times \mathbf{p}_a \tag{3}$$

is constant in time. This sum is over the body's particles, which are indexed by a, and $\mathbf{p}_a$ is the momentum corresponding to the ath particle. (The sum is an integral if the body is a continuum.)

Euler showed how to simplify the equations of motion of the free rigid body by going to a frame attached to the body. In this frame, $\mathbf{M}$ is no longer constant. We will call the angular momentum, viewed from this body-fixed frame, $\mathbf{M}_b$. Thus

$$\mathbf{M} = g\cdot\mathbf{M}_b. \tag{4}$$

In particular,

$$\|\mathbf{M}_b\|^2 = \|\mathbf{M}\|^2, \tag{5}$$

so that $\mathbf{M}_b$ moves on the surface of a sphere. The kinetic energy of the motion is

$$H = (\tfrac{1}{2})(\mathbf{M}_b\cdot\mathbb{I}_b^{-1}\mathbf{M}_b), \tag{6}$$

where $\mathbb{I}_b$ is the moment of inertia tensor of the reference body. (It is a symmetric positive definite matrix.) Both $\mathbf{M}$ and H are constants of the motion. This means that $\mathbf{M}_b$ moves along a curve defined by intersecting the sphere defined by Eq. (5) with the ellipsoid defined by Eq. (6) (see Fig. 1). Almost all of these curves are closed.

Now suppose that the constants $\mathbf{M} = \mathbf{J}$ and $H = E$ are typical, so that these curves are in fact closed. Let T be the period of $\mathbf{M}_b$'s oscillation along the curve:

$$\mathbf{M}_b(T + t_0) = \mathbf{M}_b(t_0). \tag{7}$$

Combining Eqs. (4) and (7) we see that $g(T+t_0)^{-1}\cdot\mathbf{J} = g(t_0)^{-1}\cdot\mathbf{J}$ or $g(T+t_0)g(t_0)^{-1}\cdot\mathbf{J} = \mathbf{J}$, so that $R = g(T+t_0)g(t_0)^{-1}$ is a rotation about the $\mathbf{J}$ axis. Note that $g(T+t_0) = Rg(t_0)$ so that R describes the rotation of the body in space after each orbit of its angular momentum in the body-fixed frame.

The question we pose in our title is "What is the angle $\Delta\theta$ of this rotation R?" The answer is given by Eq. (1), which we repeat:

$$\Delta\theta = 2ET/J - \Omega.$$

Here, Ω is the solid angle enclosed by the closed curve. The rest of the paper is devoted to deriving this answer.

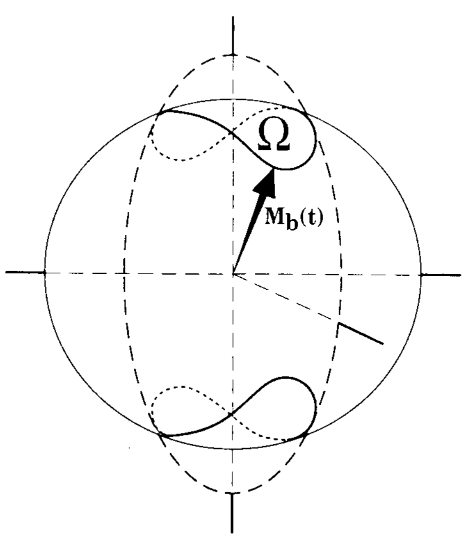

Fig. 1. The body angular momentum vector $\mathbf{M}_b(t)$ lies on the intersection of a sphere with an ellipsoid and encloses a solid angle Ω.

II. THE METHOD OF DERIVATION

A. An important curve

Suppose that $t = t_0 = 0$ corresponds to our reference configuration. Then $g(0) = I$, $\mathbf{M}_b(0) = \mathbf{J}$, and $z(0) = (I,\mathbf{J})$ are initial conditions for the motion of the rigid body. The phase-space trajectory $z(t)$ through $z(0)$ consists of pairs $[g(t),\mathbf{M}_b(t)]$. Consider the following two curves in the phase space of the rigid body, both beginning at $z(0)$ (see Fig. 2): $C_1(t) = z(t)$ for $0\leqslant t\leqslant T =$ dynamical evolution starting at $z(0)$; and $C_2(\theta) =$ counterclockwise spatial rotation of the body about the $\mathbf{J}$ axis by θ radians, $0\leqslant\theta\leqslant\Delta\theta$.

These two curves intersect when $t = T$ and $\theta = \Delta\theta$. Thus the curve $C = C_1 - C_2$ obtained by first going along C_1 and then backward along C_2 is a closed curve.

We will prove Eq. (1) by integrating the canonical one-form $p\,dq$ along C and then applying Stokes' formula to relate this line integral to a surface integral. In order to do this, we will have to learn some things about the canonical one-form.

B. The canonical one-form

What is a one-form? It is the integrand of a line integral. For example, if $\mathbf{F}$ is a (velocity-independent) force, then $\mathbf{F}\cdot d\mathbf{x}$ is the one-form whose integral along a curve is the work done in traversing that curve. An alternative definition is that a one-form is simply a differential. The differential may or may not be total, that is, it may or may not be

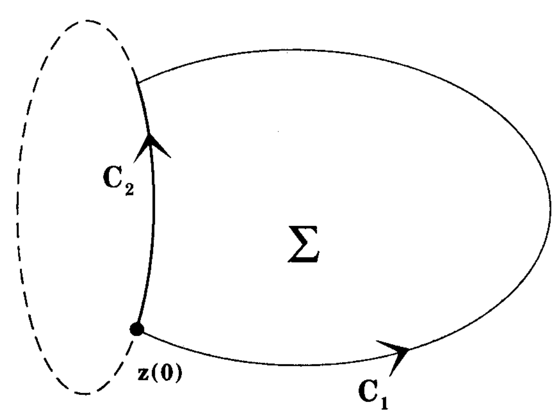

Fig. 2. A surface Σ in phase space whose boundary is the loop $C = C_1 - C_2$.

the gradient of a scalar field. The canonical one-form is not a total differential. [In tensor analysis a one-form is also referred to as a "tensor field of type (0,1)" or as a "covariant one-tensor." For further details on differential forms in general see Sec. II E and Ref. 11.]

The canonical one-form $p\,dq$ is a one-form defined on the phase space of any mechanical system. It is defined by the equation $p\,dq = \Sigma p_i\,dq_i$, where the q_i are any set of coordinates on the configuration space of the system, and the p_i are their conjugate momenta. In our particular case,

$$p\,dq = \sum \mathbf{p}_a \cdot d\mathbf{x}_a,$$

where $d\mathbf{x}_a$ denotes the deformation of the ath particle of the rigid body, and $\mathbf{p}_a$ is the momentum of this particle with respect to the center-of-mass frame. Now

$$d\mathbf{x}_a = (d\boldsymbol{\omega}) \times \mathbf{x}_a,$$

where $d\boldsymbol{\omega}$ is the infinitesimal axis of rotation, which is independent of the particle position $\mathbf{x}_a$ since the deformation is rigid. Thus

$$p\,dq = \left(\sum \mathbf{x}_a \times p_a\right) \cdot d\omega$$

or

$$p\,dq = \mathbf{M} \cdot d\omega \tag{8}$$

[compare with Eq. (3)].

We will now evaluate this line integrand for the two special types of curves that make up our curve C. This will give some physical meaning to the canonical one-form, and is a necessary step in our derivation.

The curve C_1 is parametrized by the physical time t, so that $d\boldsymbol{\omega} = \boldsymbol{\omega}\,dt$ along this curve. The angular momentum and angular velocity are related by $\mathbf{M} = \mathbb{I}\boldsymbol{\omega}$, where $\mathbb{I}$ is the moment of inertia tensor of the body with respect to the inertial frame. Thus $p\,dq = \omega \cdot \mathbb{I}\omega\,dt$ along C_1. But $\omega \cdot \mathbb{I}\omega = 2E$, where $E = H$ is the kinetic energy [compare with Eq. (6). Note that $\mathbb{I} = g\mathbb{I}_b g^{-1}$.] So

$$p\,dq = 2E\,dt \text{ along } C_1.$$

The curve C_2 is parametrized in radians, θ, so that $d\boldsymbol{\omega} = \boldsymbol{\omega}\,d\theta$ along it. Moreover, $\boldsymbol{\omega} = \mathbf{J}/J$, and $\mathbf{M} = \mathbf{J}$ on C_2. Putting these facts into Eq. (8) we get

$$p\,dq = J\,d\theta \text{ along } C_2.$$

C. Stokes' theorem and the main formula

Stokes' theorem states that

$$\oint_C p\,dq = \int\int_\Sigma d(p\,dq).$$

The double integral is a surface integral over any surface Σ whose boundary is the closed curve C and its integrand is a differential two-form (antisymmetric covariant two-tensor) called the "exterior derivative" of $p\,dq$. If Σ were to lie in a three-dimensional Euclidean space, then this Stokes' theorem would be the usual Stokes' theorem of vector analysis. However, our phase space is neither three-dimensional nor Euclidean, yet the theorem is still true. The meaning of the integrand $d(p\,dq)$ in our more general setting will become clearer below. In particular, see Sec. II E. (Again, we refer the reader interested in more details to a text which discusses differential forms, for example, Ref. 11.)

Our curve C breaks up into C_1 and C_2 so that Stokes' formula becomes

$$\int_{C_1} p\,dq - \int_{C_2} p\,dq = \int\int_\Sigma d(p\,dq),$$

where the minus sign is due to the fact that to close the loop we must travel backward along C_2 (refer to Fig. 2). We have shown that

$$p\,dq = 2E\,dt \text{ on } C_1,$$

$$p\,dq = J\,d\theta \text{ on } C_2,$$

and we will show that

$$d(p\,dq) = J\,d\Omega \text{ on } \Sigma,$$

provided Σ is contained in the three-dimensional submanifold $\mathbf{M}(q,p) = \mathbf{J}$ of our (six-dimensional) phase space. In this equation $d\Omega$ is the element of solid angle in the space of body angular momenta, and the surface Σ in phase space is to be related to the region $\Omega = \mathbf{M}_b(\Sigma)$ in the sphere by the angular momentum map $\mathbf{M}_b$.

The integrals in the equation are done immediately:

$$2ET - J\,\Delta\theta = J\Omega,$$

which upon rearrangement is our equation.

Our work then is reduced to showing that the surface integrand $d(p\,dq)$ is as just described.

D. The surface integrand

We will expand $p\,dq$ on the three-dimensional constraint surface $\mathbf{M}(q,p) = \mathbf{J}$ within phase space. The expansion will be in terms of the Euler angle coordinates (ϕ,θ,ψ) for the rotation group. We will then take the exterior derivative d of the resulting expression to obtain the two-dimensional integrand.

Euler angles are defined by

$$g(\phi,\theta,\psi) = g_3(\phi)g_2(\theta)g_3(\psi),$$

where $g_i(\theta)$ denotes the counterclockwise rotation about the ith coordinate axis by an angle of θ radians, for $i = 1,2,3$. We choose the coordinate system so that $\mathbf{J}$ is parallel to the three-axis:

$$\mathbf{J} = J\mathbf{e}_3.$$

Here, $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ is the right-handed orthonormal basis parallel whose elements are parallel to the 1-2-3 axes.

In order to express $p\,dq$ in these coordinates we will first relate the differential dg to the infinitesimal angular velocity $d\boldsymbol{\omega}$ and then plug this relation into Eq. (8) for $p\,dq$. Recall [Eq. (2)] that the matrix g relates body coordinates $\mathbf{X}$ to spatial coordinates: $\mathbf{x} = g\mathbf{X}$. Consequently,

$$d\mathbf{x} = (dg)X.$$

Earlier we noted that $d\mathbf{x} = d\boldsymbol{\omega} \times \mathbf{x}$. Therefore,

$$(dg)g^{-1}\mathbf{x} = d\boldsymbol{\omega} \times \mathbf{x}. \tag{9}$$

Let $R(\theta,\boldsymbol{\omega})$ denote the counterclockwise rotation about the $\boldsymbol{\omega}$ axis by θ rad, where $\boldsymbol{\omega}$ is fixed. Then

$$(dR)\mathbf{X} = \boldsymbol{\omega}\,d\theta \times R\,\mathbf{X}$$

or

$$(dR)R^{-1}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}. \tag{10}$$

Also $g_3(\phi) = R(\phi,\mathbf{e}_3)$, $g_2(\theta) = R(\theta,\mathbf{e}_2)$. Now differentiate g:

$$dg = [dg_3(\phi)]g_2(\theta)g_3(\psi) + g_3(\phi)[dg_2(\theta)]g_3(\psi) + g_3(\phi)g_2(\theta)[dg_3(\phi)]$$

multiply on the right by g^{-1}:

$$(dg)g^{-1} = [dg_3(\phi)]g_3(\phi)^{-1} + g_3(\phi) \times\{[dg_2(\theta)]g_2(\theta)^{-1}\}g_3(\phi)^{-1} + g_3(\phi)g_2(\theta)\{[dg_3(\psi)]g_3(\psi)^{-1}\} \times g_2(\theta)^{-1}g_3(\phi)^{-1},$$

and apply this result to $\mathbf{x}$, using Eq. (10):

$$(dg)g^{-1}\mathbf{x} = \mathbf{e}_3\, d\phi \times \mathbf{x} + g_3(\phi)[\mathbf{e}_2\, d\theta \times g_3(\phi)^{-1}\mathbf{x}] + g_3(\phi)g_2(\theta)\{\mathbf{e}_3\, d\psi \times [g_3(\phi)g_2(\theta)]^{-1}\mathbf{x}\}.$$

Now

$$g(\mathbf{v}\times\mathbf{w}) = g\mathbf{v}\times g\mathbf{w}$$

for any rotation matrix g. Thus

$$(dg)g^{-1}\mathbf{x} = [\mathbf{e}_3\, d\phi + g_3(\phi)\mathbf{e}_2\, d\theta + g_3(\phi)g_2(\theta)\mathbf{e}_3\, d\psi]\times\mathbf{x}.$$

Compare this with Eq. (9) and use the fact that $\mathbf{x}$ is arbitrary, to conclude that

$$d\boldsymbol{\omega} = \mathbf{e}_3 d\phi + g_3(\phi)\mathbf{e}_2\, d\theta + g_3(\phi)g_2(\theta)\mathbf{e}_3 d\psi.$$

Now $\mathbf{M} = \mathbf{J} = J\mathbf{e}_3$, so that according to Eq. (8),

$$p\, dq = J\{d\phi + [\mathbf{e}_3 \cdot g_2(\phi)\mathbf{e}_3]d\psi\}.$$

Here, we have used the facts that $\mathbf{e}_3$ is a unit vector, that it is perpendicular to $\mathbf{e}_2$, and that g_3 is a rotation about $\mathbf{e}_3$. Also

$$g_2(\theta)\mathbf{e}_3 = \cos(\theta)\mathbf{e}_3 + \sin(\theta)\mathbf{e}_1,$$

so we end up with

$$p\, dq = J[d\phi + \cos(\theta)d\psi].$$

E. The exterior differential and Stokes' theorem

Let q_1, q_2, q_3 be any three-dimensional coordinate system, for example, our Euler angles. A one-form is an expression of the form $\alpha = A_1\, dq_1 + A_2\, dq_2 + A_3\, dq_3$, where the A_i are functions of the q_i. Stokes' theorem asserts that

$$\oint_C \alpha = \int\int_\Sigma d\alpha, \tag{11}$$

where C is a closed curve which bounds the surface Σ and where

$$d\alpha = \left(\frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}\right)dq_1\, dq_2 + \left(\frac{\partial A_3}{\partial q_1} - \frac{\partial A_1}{\partial q_3}\right)dq_1\, dq_3 + \left(\frac{\partial A_3}{dq_2} - \frac{\partial A_2}{\partial q_3}\right)dq_2\, dq_3.$$

Stokes' theorem holds in *any* coordinate system. It does not require any mention of inner products or of normal vectors to surfaces. An orientation of the surface is required, however. (Stokes' theorem also holds in any dimension and for any degree k of differential form, although we have stated it in dimension three and for $k = 1$.) Again, the interested reader should see Ref. 11. Regardless of these generalities, the version of Stokes' theorem that we use here can be proved by noting that it has exactly the same coordinate expression as the usual Stokes' theorem of vector analysis.

In any case, for us, $\alpha = p\, dq = J(d\theta + \cos\theta\, d\psi)$, so that

$$d(p\, dq) = J(-\sin\theta\, d\theta\, d\psi) = -J\, d\Omega,$$

where $d\Omega$ is the differential element of solid angle on a sphere coordinatized by θ and ψ.

F. Finishing off the surface integrand

The Ω in our Eq. (1) is a solid angle in the space of body angular momenta. Therefore, we still must relate the $d\Omega$ of Sec. II E to the element $d\Omega_{\text{body}}$ of solid angle in the body angular momentum space.

The inertial and body angular momenta are related by $\mathbf{M} = g\mathbf{M}_b$. Therefore, on our constraint surface $\mathbf{M} = \mathbf{J}$ we have

$$\begin{aligned}\mathbf{M}_b &= g^{-1}\mathbf{J}\\ &= Jg_3(\psi)^{-1}g_2(\theta)^{-1}g_3(\phi)^{-1}\mathbf{e}_3\\ &= Jg_3(\psi)^{-1}g_2(\theta)^{-1}\mathbf{e}_3\\ &= Jg_3(\psi)^{-1}[\cos(\theta)\mathbf{e}_3 - \sin(\theta)\mathbf{e}_1]\\ &= J[\cos(\theta)\mathbf{e}_3 - \sin(\theta)g_3(\psi)^{-1}\mathbf{e}_1]\\ &= J\{\cos(\theta)\mathbf{e}_3 - \sin(\theta)[\cos(\psi)\mathbf{e}_1 - \sin(\psi)\mathbf{e}_2]\}.\end{aligned}$$

This says that our θ and ψ are related to spherical coordinates on body angular momentum space by $\psi = -\psi_{\text{body}}$, $\theta = -\theta_{\text{body}}$. This is an orientation-reversing coordinate transformation, so that $d\Omega = -d\Omega_{\text{body}}$. [In terms of Cartesian coordinates, this transformation is $(x,y,z) = (-x_{\text{body}}, y_{\text{body}}, z_{\text{body}})$.] Summarizing:

$$d(p\, dq) = J\, d\Omega_{\text{body}}.$$

This completes the demonstration of Eq. (1).

III. THE METHOD APPLIED TO OBTAIN BERRY'S PHASE

A. Formula

As another illustration of the method, we will use it to derive Berry's phase formula. Berry's phase answers the following question as reformulated by Aharanov and Anandan:[6] "What is the phase shift suffered by a quantum wave vector $|\psi\rangle$, given that its state $|\psi\rangle\langle\psi|$ has undergone a cycle?"

The structure of the calculation of this phase shift can be made identical to our calculation for the rigid body. Consider the same two curves C_1 and C_2. C_1 is defined by dynamically evolving an initial condition $|\psi_0\rangle$ according to Schrödinger's equation. C_2 is defined by rotating this initial condition: $C_2(\theta) = \exp(i\theta)|\psi_0\rangle$. The canonical one-form on Hilbert space is $p\, dq = \text{Im}\,\langle\psi|d\psi\rangle$. The expectation value of the energy defines a "classical" Hamiltonian $H(|\psi\rangle) = \langle H_{\text{op}}\rangle_\psi = :\langle\psi|H_{\text{op}}|\psi\rangle$. Here, H_{op} is the quantum Hamiltonian, a possibly time-dependent Hermitian operator needed to define Schrödinger's equation. With these choices of $p\, dq$ and H, Hamilton's equations are easily checked (and well known) to be Schrödinger's equation. Write $E(t) = H[|\psi(t)\rangle]$ for the expected energy of our particular time-dependent state. The Noether conserved quantity corresponding to the symmetry of shifting phase is $\langle\psi|\psi\rangle$. It plays the role which was played by the angular momentum in the rigid-body calculation. We assume that the initial state is normalized, that is, $\langle\psi|\psi\rangle = 1$. The two line integrands are immediately computed:

$$p\, dq = -E(t)dt \text{ on } C_1,$$

$$p\, dq = \hbar\, d\theta \text{ on } C_2.$$

The surface integrand is computed to be

$d(p\,dq) = -\hbar$(Kähler form on projective Hilbert space) on Σ. This last integrand is half of the solid angle form $d\Omega$ on the two-dimension sphere, provided we are dealing with a two-level system, so that the Hilbert space is (complex) two-dimensional. Evaluating the integrals, we arrive at the result

$$\Delta\theta = -(1/\hbar)E_{av}T - \gamma. \qquad (12)$$

The first term is the dynamic phase; T is the duration of the cycle. $E_{av} = (1/T)\int E(t)dt$ is the average energy over the cycle; and the last term γ is the geometric, or Berry's phase, and is a kind of generalized solid angle. In the particular case of a two-level system, it is exactly the solid angle. Note the similarity of Eq. (12) to our Eq. (1) upon making the substitution $\hbar = J$.

ACKNOWLEDGMENT

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