Periodic Motions of Mechanical Systems.

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§ 1.
Problem Setting.

$U(x_1, x_2, \ldots, x_n)$, which we also write in condensed form as $U(x)$, is a real, regular analytic function of the real variables $x_1, x_2, \ldots, x_n$ in a certain region $\mathfrak{G}$ of the n-dimensional number space. The points of $\mathfrak{G}$, that satisfy the inequality

$$(1) \qquad U(x) \leq E$$

for a fixed real E, should form a set $\mathfrak{E}$ that is homeomorphic to the n-dimensional ball. $\mathfrak{E}$ is the interior of $\overline{\mathfrak{E}}$. On the boundary of $\overline{\mathfrak{E}}$, thus on $\overline{\mathfrak{E}} - \mathfrak{E}$, it is then necessary that $U(x) = E$. We further require

$$(2) \qquad \operatorname{grad} U \neq 0 \quad \text{on } \overline{\mathfrak{E}} - \mathfrak{E},$$
$$(3) \qquad U < E \quad \text{on } \mathfrak{E}.$$

Furthermore

$$(4) \qquad \sum_{\rho,\sigma=1}^{n} a_{\rho\sigma}(x)\, dx_\rho dx_\sigma \qquad (a_{\rho\sigma} = a_{\sigma\rho})$$

is a positive definite quadratic form, whose coefficients $a_{\rho\sigma}(x)$ are regular analytic on $\mathfrak{G}$.

Under the given assumptions we consider a mechanical system with the Lagrangian coordinates $x_1, x_2, \ldots, x_n$, with the potential energy U and the kinetic energy

$$(5) \qquad T(x, \dot{x}) = \sum_{\rho,\sigma=1}^{n} a_{\rho\sigma}(x)\, \dot{x}_\rho \dot{x}_\sigma; \qquad \dot{x}_\rho = \frac{dx_\rho}{dt}$$

minimizing a periodic motion with the total energy E. In particular: One can find a curved arc of $\overline{\mathfrak{E}}$ that connects two points A and B of the boundary $\overline{\mathfrak{E}} - \mathfrak{E}$ and which is traversed back and forth periodically under the system.

§ 2.
The Equations of Motion.

The Lagrangian equations of motion of the system read

$$\frac{d}{dt}\frac{\partial}{\partial x_\lambda}(T-U)=\frac{\partial}{\partial x_\lambda}(T-U) \tag{6}$$

or in detail

$$\sum_{\mu=1}^{n} a_{\lambda\mu}\ddot{x}_\mu = -\frac{1}{2}\sum_{\mu,\nu=1}^{n}\left\{-\frac{\partial a_{\mu\nu}}{\partial x_\lambda}+\frac{\partial a_{\nu\lambda}}{\partial x_\mu}+\frac{\partial a_{\lambda\mu}}{\partial x_\nu}\right\}\dot{x}_\mu\dot{x}_\nu-\frac{1}{2}\frac{\partial U}{\partial x_\lambda}\,. \tag{7}$$
$$(\lambda = 1, 2, \ldots, n)$$

These equations can be solved for $\ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n$, because the determinant $|(a_{\lambda\mu})|$ is $\neq 0$. We define the functions $a^{\rho\lambda}(x)$ through the equations

$$\sum_{\lambda=1}^{n} a^{\rho\lambda}a_{\lambda\mu}=\delta^{\rho}_{\mu} \tag{8}$$

and introduce the abbreviation

$$A_{\lambda,\mu\nu}=\frac{1}{2}\left\{-\frac{\partial a_{\mu\nu}}{\partial x_\lambda}+\frac{\partial a_{\nu\lambda}}{\partial x_\mu}+\frac{\partial a_{\lambda\mu}}{\partial x_\nu}\right\}. \tag{9}$$

Then we have

$$\ddot{x}_\rho = -\sum_{\lambda,\mu,\nu=1}^{n} a^{\rho\lambda}A_{\lambda,\mu\nu}\dot{x}_\mu\dot{x}_\nu-\frac{1}{2}\sum_{\lambda=1}^{n}a^{\rho\lambda}\frac{\partial U}{\partial x_\lambda}\,. \tag{10}$$

One can reduce this system of n second-order differential equations to a system of $2n$ first-order differential equations, by setting

$$z_\mu = \dot{x}_\mu\,. \tag{11}$$

The first-order system then reads

$$\begin{aligned}\dot{x}_\rho &= z_\rho\\ \dot{z}_\rho &= -\sum_{\lambda,\mu,\nu=1}^{n} a^{\rho\lambda}A_{\lambda,\mu\nu}z_\mu z_\nu-\frac{1}{2}\sum_{\lambda=1}^{n}a^{\rho\lambda}\frac{\partial U}{\partial x_\lambda}\,.\end{aligned} \tag{12}$$

The right sides are regular functions of $x_1, x_2, \ldots, x_n; z_1, z_2, \ldots, z_n; t$ in a certain region $\mathfrak{H}$ of the $(2n+1)$-dimensional space, that one obtains when $x_1, x_2, \ldots, x_n$ are in $\mathfrak{G}$ and $z_1, z_2, \ldots, z_n; t$ are allowed to vary as you please. Moreover, t does not generally appear in the right sides of (12) .

If $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n; \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n; \bar{t}$ is a point in $\mathfrak{H}$, then the system (12) has an exact solution

$$x_\rho = \varphi_\rho(t)\,, \quad z_\rho = \psi_\rho(t)\,, \qquad (\rho = 1, 2, \ldots, n) \tag{13}$$

for which the initial conditions

$$\bar{x}_\rho = \varphi_\rho(\bar{t})\,, \quad \bar{z}_\rho = \psi_\rho(\bar{t}) \tag{14}$$

hold, and one can extend the solution on both sides as far as the boundary of $\mathfrak{H}$.[1] φ_ρ and ψ_ρ are regular functions of t.

Now, as is well-known, the total energy $T + U$ is constant during the course of the motion. We consider in particular a motion having the total energy E, so we have

$$U(x) + T(x, z) = E, \tag{15}$$

thus because $T(x, z) \geq 0$ it follows that

$$U(x) \leq E\,. \tag{16}$$

This inequality says that the point x is constrained to lie in the closed domain $\overline{\mathfrak{E}}$.

Hence it now follows further, that in the course of the entire motion the velocity components z_ρ are bounded. Then we have

$$T(x, z) = E - U(x)\,,$$

and the right side is a continuous and hence bounded function of x on $\overline{\mathfrak{E}}$. Since $T(x, z)$ is positive definite, it follows then, that the velocity components z_ρ must be bounded.[2] Thus the solution (13) comes near the boundary of $\mathfrak{H}$ for no finite value of t.

[1] Vgl. E. KAMKE, Differentialgleichungen reeler Funktionen (Leipzig 1945), p. 135.

[2] $\sum_{\rho,\sigma=1}^n a_{\rho\sigma}(x)\, Z_\rho Z_\sigma$ is a continuous function of x and Z, whenever the point x is in $\overline{\mathfrak{F}}$ and Z varies on the unit sphere $\sum_{\rho=1}^n Z_\rho^2 = 1$. This function thus takes a minimum, which must be positive due to the positive definiteness of the quadratic form:

$$0 < m \leq \sum_{\rho,\sigma=1}^{n} a_{\rho\sigma}(x)\, Z_\rho Z_\sigma\,.$$

If we now let $z_\rho = \lambda Z_\rho$, then it follows that

The motion of the mechanical system with the total energy E is hence, through its initial position and initial velocity, defined in the entire interval $-\infty < t < +\infty$.

Through the first n equations (13) we obtain a curve $x_\rho = \varphi_\rho(t)$, $(\rho = 1, 2, \ldots, n)$ which we call an *orbit* of the system.

§ 3.

Introduction of New Coordinates in the Neighborhood of a Boundary Point.

Let O be a boundary point of $\mathfrak{E}$. To investigate the orbits in the neighborhood of O, we introduce an affine coordinate system $\xi_1, \xi_2, \ldots, \xi_n$ with the origin at O, so that at the point O we have

$$\sum_{\rho,\sigma=1}^{n} a_{\rho\sigma} dx_\rho dx_\sigma = \sum_{\rho=1}^{n} d\xi_\rho^2, \tag{17}$$

$$\frac{\partial U}{\partial \xi_r} = 0, \qquad (r = 1, 2, \ldots, n-1) \tag{18}$$

$$\frac{\partial U}{\partial \xi_n} < 0. \tag{19}$$

The ξ-system is determined up to an orthogonal transformation of $\xi_1, \xi_2, \ldots, \xi_{n-1}$.

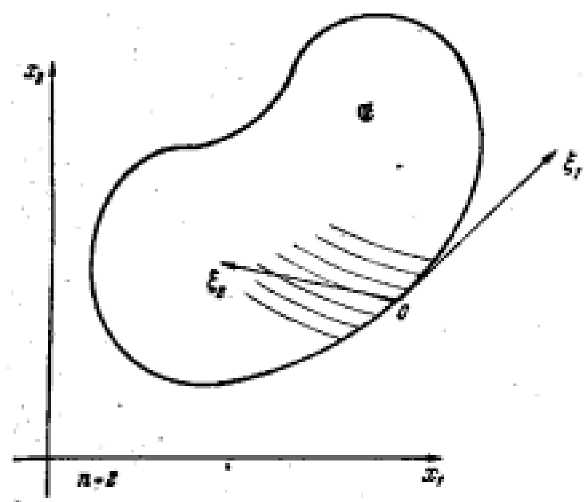

Fig. 1.

$$m\lambda^2 \leq \sum_{\rho,\sigma=1}^{n} a_{\rho\sigma}(x) z_\rho z_\sigma$$

or

$$m\lambda^2 \leq E - U,$$

therefore

$$\sum z_\rho^2 == \lambda^2 \leq \frac{E-U}{m}.$$

We write the quadratic form (4), transformed into the ξ-system, in general as

$$\sum_{\rho,\sigma=1}^{n} a_{\rho\sigma}(x)\, dx_\rho dx_\sigma = \sum_{\rho,\sigma=1}^{n} \alpha_{\rho\sigma}(\xi)\, d\xi_\rho d\xi_\sigma, \tag{20}$$

so that (17) is equivalent to

$$\alpha_{\rho\sigma}(0) = \delta_{\rho\sigma}\,. \tag{21}$$

Coordinate-transformed into the ξ-system, the invariant equations of motion corresponding to § 2 (10) read

$$\ddot{\xi}_\rho = -\sum_{\lambda,\mu,\nu} \alpha^{\rho\lambda} A_{\lambda,\mu\nu} \dot{\xi}_\mu \dot{\xi}_\nu - \frac{1}{2}\sum_{\lambda} \alpha^{\rho\lambda} \frac{\partial U}{\partial \xi_\lambda}, \qquad (\rho = 1, 2, \dots, n) \tag{22}$$

wherein $\alpha^{\rho\lambda}$ and $A_{\lambda,\mu\nu}$ are defined through the equations

$$\sum_{\lambda=1}^{n} \alpha^{\rho\lambda} \alpha_{\lambda\mu} = \delta^{\rho}_{\mu}, \tag{23}$$

$$A_{\lambda,\mu\nu} = \frac{1}{2}\left\{ -\frac{\partial \alpha_{\mu\nu}}{\partial \xi_\lambda} + \frac{\partial \alpha_{\nu\lambda}}{\partial \xi_\mu} + \frac{\partial \alpha_{\lambda\mu}}{\partial \xi_\nu} \right\}. \tag{24}$$

Following the physical chemistry practice, we denote the potential energy in all coordinate systems by the same capital letter U.

While, above all, we are obligated to respect the claim that the total energy of the system refrains from decreasing, we consider particular solutions to (22), those for which

$$\dot{\xi}_\rho = 0 \qquad (\rho = 1, 2, \dots, n)$$

at the time $t = 0$; thus, those solutions that begin with null velocity. At each point $\bar{\xi}_\rho$ in the neighborhood of O (with respect to $\mathfrak{G}$), there exists a unique orbit that starts there with null velocity:

$$\xi_\rho = F_\rho\left(t, \bar{\xi}\right), \tag{25}$$

$$\bar{\xi}_\rho = F_\rho\left(0, \bar{\xi}\right). \tag{26}$$

According to existence theorems about solutions of systems of differential equations, the functions F_ρ are analytic in $t, \bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n$ in the neighborhood of $t = \bar{\xi}_1 = \bar{\xi}_2 = \cdots = \bar{\xi}_n = 0$.

In addition to (25),

$$\xi_\rho = F_\rho\left(-t, \bar{\xi}\right) \tag{27}$$

is also a solution of (22), since (22) does not change under the substitution of $-t$ for t. But since (25) and (27) satisfy the same initial conditions, both solutions must be identical:

$$F_\rho\left(t,\bar{\xi}\right) = F_\rho\left(-t,\bar{\xi}\right) .$$

In the power series expansion of F_ρ, only the even powers of t occur; that is, ξ_ρ is a regular function of $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n$ and $\tau = t^2$:

$$\xi_\rho = f_\rho\left(\tau,\bar{\xi}\right) , \tag{28}$$

$$\bar{\xi}_\rho = f_\rho\left(0,\bar{\xi}\right) . \tag{29}$$

Accordingly, we write the power series as

$$\xi_\rho = \bar{\xi}_\rho + c_\rho\left(\bar{\xi}\right)\tau + \cdots .$$

Applying these initial terms to (22) yields

$$2c_\rho = -\tfrac{1}{2}\sum_{\lambda=1}^{n} \alpha^{\rho\lambda}\left(\bar{\xi}\right) U_\lambda\left(\bar{\xi}\right) ,$$

where we have used the abbreviation

$$U_\lambda\left(\xi\right) = \frac{\partial U\left(\xi\right)}{\partial \xi_\lambda} . \tag{30}$$

Thus the expansion reads

$$\xi_\rho = \bar{\xi}_\rho - \tfrac{1}{4}\sum_{\lambda=1}^{n} \alpha^{\rho\lambda}\left(\bar{\xi}\right) U_\lambda\left(\bar{\xi}\right)\tau + \cdots . \tag{31}$$

In particular, we consider the orbits that begin on the boundary of $\mathfrak{E}$. For them the total energy is E. The coordinates $\bar{\xi}_\rho$ of the initial points are then connected with one another by the equation

$$U\left(\bar{\xi}\right) = E . \tag{32}$$

Since $U\left(0\right) = E$ and

$$U_r\left(0\right) = 0 \qquad\qquad (r = 1, 2, \ldots, n-1)$$

holds, $\bar{\xi}_n$ in the neighborhood of O is allowed the expansion

$$\bar{\xi}_n = \psi\left(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}\right) , \tag{33}$$

whereby in the power series ψ the constant term and the linear terms are missing.

If $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}, \tau$ are given, then, provided the values of these dimensions are sufficiently small, one can express $\bar{\xi}_n$ from (33) and thereafter ξ_ρ from (31) as regular functions of $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}, \tau$. In consequence of (31), (21), and (18), at the point O we have

$$\begin{aligned} d\xi_r &= d\bar{\xi}_r, \qquad (r = 1, 2, \ldots, n-1) \\ d\xi_n &= -\tfrac{1}{4} U_n(0)\, d\tau, \end{aligned} \tag{34}$$

so that the transition from $\xi_1, \xi_2, \ldots, \xi_n$ to $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}, \tau$ in the neighborhood of O is an invertible regular analytic coordinate transformation. The new coordinate system has the advantage that the coordinate lines are orbits emanating from the boundary of $\mathfrak{E}$. However it is defined only in a certain neighborhood of O.

§ 4.
Introduction of the Riemannian Metric.

As is well-known, the problem is to find the orbits with the total energy E, or equivalently, to determine the geodesic lines of the Riemannian manifold, which the region $\mathfrak{E}$ becomes through the imposition of the metric[3]

$$ds^2 = (E - U) \sum_{\rho,\sigma=1}^{n} a_{\rho\sigma}(x)\, dx_\rho dx_\sigma\,. \tag{35}$$

But observe that the boundary of $\mathfrak{E}$ does not belong to the Riemannian manifold, because $ds^2 = 0$ on the boundary. Accordingly, the geodesics of the Riemannian manifold are defined only on $\mathfrak{E}$. A geodesic curve situated in $\mathfrak{E}$ is also a part of an orbit. However, an orbit in its course can also contain points of the boundary $\bar{\mathfrak{E}} - \mathfrak{E}$, so it is not necessarily the case that it is a geodesic.

In the coordinates $\xi_1, \xi_2, \ldots, \xi_n$ the bilinear form reads

$$ds^2 = (E - U) \sum_{\rho,\sigma=1}^{n} \alpha_{\rho\sigma}(\xi)\, d\xi_\rho d\xi_\sigma\,. \tag{36}$$

Denote by S the arc length of the orbit $\bar{\xi}_1 = \text{const.}, \bar{\xi}_2 = \text{const.}, \ldots, \bar{\xi}_{n-1} = \text{const.}$, increasing with increasing τ; hence we have

$$\left(\frac{dS}{d\tau}\right)^2 = (E - U) \sum_{\rho,\sigma=1}^{n} \alpha_{\rho\sigma}(\xi) \frac{\partial \xi_\rho}{\partial \tau} \frac{\partial \xi_\sigma}{\partial \tau}, \tag{37}$$

[3] C. Carathéodory, Variationsrechnung und die partiellen Differentialgleichungen erster Ordnung (Leipzig 1935), §§ 308 and 309.

wherein ξ_λ and $\frac{\partial \xi_\lambda}{\partial \tau}$ are defined by (31). Accordingly, $\left(\frac{dS}{d\tau}\right)^2$ is a regular function of $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}, \tau$, vanishing for $\tau = 0$:

$$\left(\frac{dS}{d\tau}\right)^2 = \tau \mathfrak{B}\left(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}; \tau\right) . \tag{38}$$

We determine the constant term in the power series $\mathfrak{B}$. According to (21) it is

$$\alpha_{\rho\sigma}(0) = \delta_{\rho\sigma}$$

and, because of (31) and (18),

$$\left[\frac{\partial \xi_\rho}{\partial \tau}\right]_0 = -\tfrac{1}{4} \sum_{\lambda=1}^{n} \delta_{\rho\lambda} U_\lambda(0) = -\tfrac{1}{4} \delta_{\rho n} U_n(0) ; \tag{39}$$

where the function argument (0) signifies that we are setting $\bar{\xi}_1 = \cdots = \bar{\xi}_{n-1} = \tau = 0$. Furthermore, we have

$$\frac{\partial U\left(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}; \tau\right)}{\partial \tau} = \sum_{\rho=1}^{n} U_\rho \frac{\partial \xi_\rho}{\partial \tau},$$

thus

$$\left[\frac{\partial U\left(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}; \tau\right)}{\partial \tau}\right]_0 = -\tfrac{1}{4} U_n^2(0) \tag{40}$$

or

$$\lim_{\bar{\xi}_1, \ldots, \bar{\xi}_{n-1}, \tau \to 0} \frac{E - U}{\tau} = -\left[\frac{\partial U}{\partial \tau}\right]_0 = \tfrac{1}{4} U_n^2(0) . \tag{41}$$

Consequently we obtain

$$\mathfrak{B}(0, \ldots; 0) = \tfrac{1}{4} U_n^2(0) \cdot \left(\tfrac{1}{4} U_n(0)\right)^2 = \tfrac{1}{4^3} U_n^4(0) ,$$

and therewith

$$\frac{dS}{d\tau} = \sqrt{\tau} \mathfrak{B}_1\left(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}; \tau\right) ; \qquad \mathfrak{B}_1(0, \ldots; 0) = \tfrac{1}{2^3} U_n^2(0) .$$

Taking the root with the positive sign, S must increase with τ, as $\tau \geq 0$ is increasing. Hence, through integration it follows that

$$S = \tau^{\frac{3}{2}} \mathfrak{B}_2\left(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}; \tau\right) ; \qquad \mathfrak{B}_2(0, \ldots; 0) = \tfrac{1}{12} U_n^2(0) . \tag{42}$$

The constant of integration has been chosen so that for $\tau = 0$, S vanishes on the boundary of $\mathfrak{E}$. From (42) it follows that

$$(43) \qquad S^{\frac{2}{3}} = \tau \mathfrak{B}_3\left(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}; \tau\right); \qquad \mathfrak{B}_3(0, \ldots; 0) = \left(\tfrac{1}{12}\right)^{\frac{2}{3}} U_n^{\frac{4}{3}}(0) .$$

Hence $S^{\frac{2}{3}}$ is a regular function of $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}; \tau$ in the neighborhood of O.

We replace $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}; \tau$ with the new independent variables $y_1, y_2, \ldots, y_n$, by means of the transformation

$$(44) \qquad \begin{aligned} y_r &= \bar{\xi}_r, \qquad\qquad\qquad (r = 1, 2, \ldots, n-1) \\ y_n &= S^{\frac{2}{3}} = \tau \mathfrak{B}_3\left(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}; \tau\right) . \end{aligned}$$

At the point O one has

$$(45) \qquad \begin{aligned} dy_r &= d\bar{\xi}_r, \qquad\qquad\qquad (r = 1, 2, \ldots, n-1) \\ dy_n &= \left(\tfrac{1}{12}\right)^{\frac{2}{3}} U_n^{\frac{4}{3}}(0)\, d\tau . \end{aligned}$$

Consequently, an invertible regular transformation exists in the neighborhood of O.

Like $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_{n-1}, \tau$, the coordinates $y_1, y_2, \ldots, y_n$ have the property that the parameter lines, namely $y_1 = \text{const.}, y_2 = \text{const.}, \ldots, y_{n-1} = \text{const.}$, are orbits emanating from the boundary of $\mathfrak{E}$. There y_n is a function of the arc length S, thus $y_n = \text{const.}$ can be interpreted as a parallel surface to the boundary of $\mathfrak{E}$.

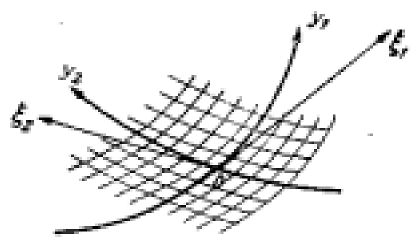

Fig. 2.

We shall see, that the surfaces $y_n = \text{const.}$ (> 0) stay perpendicular to the coordinate lines $y_1 = \text{const.}, y_2 = \text{const.}, \ldots, y_{n-1} = \text{const.}$ Moreover, we note the following theorem of differential geometry:

If

$$y_\rho = y_\rho(S, \lambda) \qquad\qquad (\rho = 1, 2, \ldots, n)$$

are analytic functions of S and λ and, for a fixed value of the parameter λ, the equations describe a geodesic with the arc length S, then the orthogonal projection of the vectors $\frac{\partial y_\rho}{\partial \lambda}$ on the geodesics passing through their base points (that is, on their tangents) is independent of S.

In our case, we set

$$
\begin{aligned}
y_r &= y_r(\lambda) \qquad\qquad (r = 1, 2, \ldots, n-1) \\
y_n &= y_n(S) \left(= S^{\frac{2}{3}}\right),
\end{aligned}
$$

so according to this theorem, the orthogonal projection of the vectors $\left(\frac{\partial y_1}{\partial \lambda}, \frac{\partial y_2}{\partial \lambda}, \ldots, \frac{\partial y_{n-1}}{\partial \lambda}, 0\right)$ on the geodesic $y_r = y_r(\lambda)$ $(r = 1, 2, \ldots, n-1)$ is independent of S. But as $S \to 0$, the length of the vectors, thus also of their orthogonal projections, goes to zero. That says: a vector in the $y_1, y_2, \ldots, y_n$-system, whose n^{th} component vanishes, stays perpendicular to the geodesic $y_r =$ const. that passes through its origin.

According to this, in the coordinates y_ρ the bilinear form reads

$$
\begin{aligned}
ds^2 &= (E-U) \sum_{\rho,\sigma=1}^{n} a_{\rho\sigma} dx_\rho dx_\sigma = (E-U) \sum_{\rho,\sigma=1}^{n} b_{\rho\sigma} dy_\rho dy_\sigma \\
&= (E-U) \left\{ \sum_{r,s=1}^{n-1} b_{rs} dy_r dy_s + b_{nn} dy_n^2 \right\}. \qquad (b_{\rho\sigma} = b_{\sigma\rho})
\end{aligned} \tag{46}
$$

Since

$$
\frac{\partial y_n}{\partial S} = \frac{2}{3} S^{-\frac{1}{3}}
$$

must hold for $dy_1 = dy_2 = \cdots = dy_{n-1} = 0$, it follows that b_{nn} can be calculated as

$$
b_{nn} = \frac{9}{4(E-U)} y_n \,. \tag{47}
$$

For the b_{rs} we have

$$
b_{rs}(0) = \delta_{rs} \,. \qquad (r, s = 1, 2, \ldots, n-1) \tag{48}
$$

Then in general

$$
\sum_{\rho,\sigma=1}^{n} \alpha_{\rho\sigma} d\xi_\rho d\xi_\sigma = \sum_{\rho,\sigma=1}^{n} b_{\rho\sigma} dy_\rho dy_\sigma
$$

and at the point O

$$
\begin{aligned}
d\xi_r &= dy_r \qquad\qquad (r = 1, 2, \ldots, n-1) \\
d\xi_n &= \text{const. } dy_n
\end{aligned}
$$

and

$$\alpha_{\rho\sigma}(0) = \delta_{\rho\sigma} .$$

We refer to $y_1, y_2, \ldots, y_n$ as a *normal coordinate system* pertaining to the point O.

In view of the Heine-Borel Covering Theorem, one can now cover the boundary of $\mathfrak{E}$ with finitely many normal coordinate systems pertaining to certain points $O_1, O_2, \ldots, O_m$. The normal coordinate system pertaining to the point O_μ is

$$y_{1\mu}, y_{2\mu}, \ldots, y_{n\mu} . \tag{49}$$

There these coordinate systems are related to $x_1, x_2, \ldots, x_n$ through invertible, regular analytic transformations; thus two normal coordinate systems, (49) and $y_{1\lambda}, y_{2\lambda}, \ldots, y_{n\lambda}$, are also related through such a transformation, provided the intersection of their regions of definition is not empty. — Note that $y_{n\mu}$ as well as $y_{n\lambda}$ is equal to $S^{\frac{2}{3}}$ in $\mathfrak{E}$; thus $y_{n\mu}$ and $y_{n\lambda}$ are in agreement. We can therefore henceforth suppress the second index of y_n. y_n is a regular analytic function in the neighborhood of the boundary of $\mathfrak{E}$.

If the point set $|y_n| < \varepsilon$ can be covered with finitely many normal coordinate systems — for sufficiently small ε that is certainly the case — then we call it a *normal neighborhood of the boundary of* $\mathfrak{E}$. In the following we will always understand $y_1, y_2, \ldots, y_n$ to be a normal coordinate system pertaining to a boundary point O.

§ 5.
Behavior of the Orbits in the Neighborhood of the Boundary.

Concerning the behavior of the orbits in the neighborhood of the boundary, the following holds:

Theorem 1. *For a given $\varepsilon > 0$ there is a $\delta > 0$ and a normal neighborhood $\mathfrak{U}$ of the boundary of E, in which for each orbit with total energy E we have*

$$|\dot{y}_n| < \varepsilon, \qquad \ddot{y}_n > \delta . \tag{50}$$

Proof. We consider a normal coordinate system $y_1, y_2, \ldots, y_n$ pertaining to a boundary point O. There exists a sufficiently small neighborhood of O, such that for each orbit with total energy E the inequalities

$$|\dot{y}_\rho| < \varepsilon . \qquad (\rho = 1, 2, \ldots, n) \tag{51}$$

are satisfied therein. Hence it follows from this that

$$T = \sum_{\rho,\sigma=1}^{n} b_{\rho\sigma} \dot{y}_\rho \dot{y}_\sigma = E - U$$

goes to zero as $y_\rho \to 0$, since $(b_{\rho\sigma}(0))$ is the matrix of a positive definite quadratic form (see footnote [2] on page 3).

To demonstrate the inequality $\ddot{y}_n > \delta$ we use the differential equations of the motion written in the coordinates $y_1, y_2, \ldots, y_n$. Since the equations of motion are invariant under coordinate transformations, equation (10) is thus preserved when one replaces x_ρ with y_ρ, $a_{\rho\sigma}$ with $b_{\rho\sigma}$, $a^{\rho\sigma}$ with $b^{\rho\sigma}$, and $A_{\lambda,\mu\nu}$ with $B_{\lambda,\mu\nu}$; where $b^{\rho\sigma}$ and $B_{\lambda,\mu\nu}$ are defined through the equations

$$\sum_\sigma b^{\rho\sigma} b_{\sigma\lambda} = \delta_{\rho\lambda}, \quad B_{\lambda,\mu\nu} = \frac{1}{2}\left(-\frac{\partial b_{\mu\nu}}{\partial y_\lambda} + \frac{\partial b_{\nu\lambda}}{\partial y_\mu} + \frac{\partial b_{\lambda\mu}}{\partial y_\nu}\right).$$

Thus it holds that

$$\ddot{y}_n = -\sum_{\lambda,\mu,\nu} b^{n\lambda} B_{\lambda,\mu\nu} \dot{y}_\mu \dot{y}_\nu - \frac{1}{2}\sum_\lambda b^{n\lambda}\frac{\partial U}{\partial y_\lambda}. \tag{52}$$

Note that $b^{n\lambda} B_{\lambda,\mu\nu}$ is regular in the neighborhood of O. In consequence of (51), the first term on the right side of (52) thus becomes arbitrarily small in value, provided only that one makes the neighborhood sufficiently small. — At the point O the second term in (52) has the value — see (47) —

$$-\frac{1}{2b_{nn}}\frac{\partial U}{\partial y_n} = \frac{2}{9}\left(\frac{\partial U}{\partial y_n}\right)^2 > 0;$$

the strict inequality > 0 holds here (and not ≥ 0), because according to (2) we have $\operatorname{grad} U \neq 0$ at the point O. Therefore the right side of (52) has a positive lower limit in the neighborhood of O. For each boundary point O, one can construct a neighborhood in which $|\dot{y}_n| < \varepsilon$ and the lower limit of $\ddot{y}_n$ is > 0; thus Theorem 1 is proved by applying the Heine-Borel Covering Theorem.

A corollary of Theorem 1 is

Theorem 2. *Let $\mathfrak{U}$ be a normal neighborhood as in Theorem 1 and let $x_\rho = x_\rho(t)$ be an orbit with total energy E that lies in $\mathfrak{U}$ for $t_1 \leq t \leq t_2$. Then*

$$t_2 - t_1 < \frac{2\varepsilon}{\delta}. \tag{53}$$

Thus the point described by the orbit cannot dwell in $\mathfrak{U}$ for a unit of time longer than $2\varepsilon/\delta$.

Proof. From the second inequality in (50) it follows that

$$\dot{y}_n(t_2) - \dot{y}_n(t_1) = \int_{t_1}^{t_2} \ddot{y}_n(t)\,dt > \int_{t_1}^{t_2} \delta dt = \delta \cdot (t_2 - t_1).$$

The left side is thus positive, and from the first inequality in (50) one has

$$2\varepsilon > \dot{y}_n(t_2) - \dot{y}_n(t_1).$$

Hence equation (53) follows.

Now let $x_\rho = x_\rho(t)$ $(\rho = 1, 2, \ldots, n)$ be a fixed orbit with the total energy E $(-\infty < t < +\infty)$. We can introduce the arc length s on it, whereby we let the s-value increase corresponding to increasing t-values. We assert:

Theorem 3. *s increases monotonically from $-\infty$ to $+\infty$ as t traverses the interval $-\infty$ to $+\infty$.*

For proof, let $\mathfrak{U}$ be the neighborhood of Theorem 1, described perhaps through the inequality

$$|y_n| < \eta\,.$$

We consider the smaller neighborhood $\mathfrak{B}$ described by

$$|y_n| < \frac{\eta}{2}\,.$$

It follows from (35) and (15) that

$$\left(\frac{ds}{dt}\right)^2 = (E-U)\,T\,(x,\dot{x}) = (E-U)^2\ .$$

On $\mathfrak{E} - \mathfrak{B}$ now, $E - U$ is larger than a positive number m. Therefore, as long as the orbit passes through $\mathfrak{E} - \mathfrak{B}$, one has

$$\int ds > m \int dt\,. \tag{54}$$

Thus the assertion of this theorem is known, if the orbit passes through $\mathfrak{E} - \mathfrak{B}$ persistently.

On the other hand, if the curve penetrates into $\mathfrak{B}$, then it must previously penetrate into $\mathfrak{U}$, and there is a maximum time interval for which $y_n(t) < \eta$ holds, perhaps the interval $t_1 \leq t \leq t_2$ (Fig. 3). According to Theorem 2, we have $t_2 - t_1 < \frac{2\varepsilon}{\delta}$. Because $\ddot{y}_n > \delta$, the curve $y_n = y_n(t)$ turns its convex face to the t-axis for this interval. Therefore there is a certain interval $t_1' \leq t \leq t_2'$ $(t_1 < t_1' < t_2' < t_2)$, wherein $y_n(t) < \frac{\eta}{2}$ holds; here we have used the assumption that the curve penetrates into $\mathfrak{B}$.

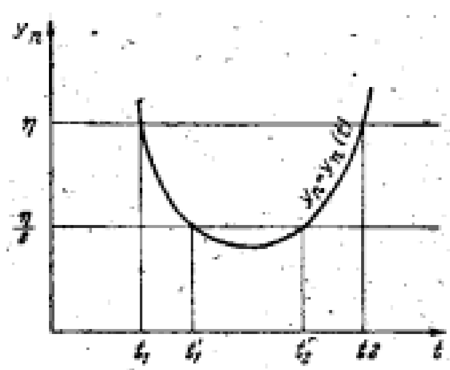

Fig. 3.

Now if the orbit penetrates into $\mathfrak{B}$ only finitely often for positive t, then it lingers in $\mathfrak{B}$ for only a finite time. From inequality (54) it follows that $\lim\limits_{t\to+\infty} s = +\infty$. On the other hand, if the orbit penetrates into $\mathfrak{B}$ infinitely often for positive t, then it also traverses the bowl $\frac{\eta}{2} \leq y_n \leq \eta$ infinitely often; whereby, according to the last equation in (44), s always increases by at least

$$\eta^{\frac{3}{2}} - \left(\frac{\eta}{2}\right)^{\frac{3}{2}}$$

on each traverse. Thus it likewise follows that $\lim\limits_{t\to+\infty} s = +\infty$. A corresponding conclusion holds for $t < 0$.

Now let g be a geodesic curve of finite length in $\mathfrak{E}$, that is not contained in a longer geodesic curve. On the pertinent orbit $x_\rho = \varphi_\rho(t)$, a finite time interval, perhaps $T_1 \leq t \leq T_2$, complies with Theorem 3. $\varphi_\rho(T_1)$ and $\varphi_\rho(T_2)$ must necessarily be points of the boundary $\overline{\mathfrak{E}} - \mathfrak{E}$, otherwise one could lengthen g there. Thus the orbit persists, traversing back and forth infinitely often, following its destiny, locked down to the geodesic curve g.

§ 6.
Geodesic Convexity.

Let $\mathfrak{M}^{n-1}$ be an analytic $(n-1)$-dimensional surface in an analytic n-dimensional Riemannian manifold $\mathfrak{M}^n$ with the Gaussian coordinates $y_1, y_2, \ldots, y_n$. We take it to be a coordinate surface $y_n = c = \text{const.}$ and choose the coordinates so that the coordinate lines $y_1 = \text{const.}, y_2 = \text{const.}, \ldots, y_{n-1} = \text{const.}$ cut the surface perpendicularly. That is, the bilinear form

$$ds^2 = \sum_{\rho,\sigma=1}^{n} g_{\rho\sigma} dx_\rho dx_\sigma \tag{55}$$

is obliged to satisfy the conditions

$$g_{rn} = 0 \ \ (r = 1, 2, \ldots, n-1) \ \text{ for } y_n = c\,. \tag{56}$$

Then we say that the surface is *geodesically convex in the direction of increasing y_n at the point P*, if for each geodesic $y_\rho = y_\rho(s)$ that comes into contact with the point P, we have

$$y_n'' = \frac{d^2 y_n}{ds^2} > 0\,. \tag{57}$$

This condition is independent of the choice of the coordinate system. That is, if $z_1, z_2, \ldots, z_n$ is another coordinate system, in which the surface is defined through the equation $z_n = d$ and whose coordinate lines $z_1 = \text{const.}, z_2 = \text{const.}, \ldots, z_{n-1} = \text{const.}$ cut the surface perpendicularly, with z_n increasing in the same direction as y_n increases, then we have

$$
\begin{aligned}
y'_n &= \sum_{\rho=1}^{n} \frac{\partial y_n}{\partial z_\rho} z'_\rho, \\
y''_n &= \sum_{\rho,\sigma=1}^{n} \frac{\partial^2 y_n}{\partial z_\rho \partial z_\sigma} z'_\rho z'_\sigma + \sum_{\rho=1}^{n} \frac{\partial y_n}{\partial z_\rho} z''_\rho .
\end{aligned} \tag{58}
$$

Since the geodesics of the surface meet at the point P, we have $z'_n = 0$ at P. Furthermore, $y_n - c$ can be expanded in powers of $z_n - d$ as follows:

$$
y_n - c = (z_n - d)\, k_1 (z_1, z_2, \ldots, z_{n-1}) + \cdots .
$$

Thus the derivatives of arbitrary order of y_n with respect to $z_1, z_2, \ldots, z_{n-1}$ are zero on the surface. Therewith (58) reduces to

$$
y''_n = \frac{\partial y_n}{\partial z_n} z''_n . \tag{59}
$$

Now since z_n is required to increase in the same direction as y_n, we have $\frac{\partial y_n}{\partial z_n} > 0$; and it follows that (57) is equivalent to $z''_n > 0$, as was to be shown.

One can readily express the condition (57) by means of the fundamental values $g_{\rho\sigma}$ of the Riemannian metric of M^n. The differential equations of the geodesics[4] read

$$
y''_\lambda = - \sum_{\mu,\nu=1}^{n} \Gamma^\lambda_{\mu\nu} y'_\mu y'_\nu ,
$$

where

$$
\Gamma^\lambda_{\mu\nu} = \sum_{\rho=1}^{n} g^{\lambda\rho} \Gamma_{\rho,\mu\nu}
$$

and

$$
\Gamma_{\lambda,\mu\nu} = \frac{1}{2} \left(- \frac{\partial g_{\mu\nu}}{\partial y_\lambda} + \frac{\partial g_{\nu\lambda}}{\partial y_\mu} + \frac{\partial g_{\lambda\mu}}{\partial y_\nu} \right) .
$$

In particular, for $\lambda = n$ we have

$$
y''_n = - \sum_{\mu,\nu=1}^{n} \Gamma^n_{\mu\nu} y'_\mu y'_\nu .
$$

Inasmuch as $y'_n = 0$ at the contact point P, it follows that

$$
y''_n = - \sum_{r,s=1}^{n-1} \Gamma^n_{rs} y'_r y'_s .
$$

[4]C. Carathéodory, loc. cit., § 310.

Now because of (56) we have

$$\Gamma^n_{rs} = \sum_{\lambda=1}^{n} g^{n\lambda}\Gamma_{\lambda,rs} = g^{nn}\Gamma_{n,rs} = \frac{1}{g_{nn}}\Gamma_{n,rs}$$

and

$$\Gamma_{n,rs} = \frac{1}{2}\left(-\frac{\partial g_{rs}}{\partial y_n} + 0 + 0\right) .$$

Thus one obtains in its entirety

$$y_n'' = \frac{1}{2g_{nn}} \sum_{r,s=1}^{n-1} \frac{\partial g_{rs}}{\partial y_n} y_r' y_s' . \tag{60}$$

Hence the inequality (57) is satisfied for all geodesics touching at P, if and only if

$$\sum_{r,s=1}^{n-1} \frac{\partial g_{rs}}{\partial y_n} y_r' y_s' \tag{61}$$

is a positive definite quadratic form at the point P.

In the case of our problem, it follows from (46) that

$$g_{rs} = (E - U)\, b_{rs},$$

thus, according to (48), at the boundary point O we have

$$\left[\frac{\partial g_{rs}}{\partial y_n}\right]_0 = -\left[\frac{\partial U\,(y)}{\partial y_n}\right]_0 \delta_{rs} .$$

Now at the point O, $\frac{\partial U(y)}{\partial y_n}$ and $\frac{\partial U(\xi)}{\partial \xi_n}$ differ from each other only by a positive factor (according to (34) and (45)) and there we have $\frac{\partial U(\xi)}{\partial \xi_n} < 0$, according to (19); thus (61) is positive definite in a small enough neighborhood of $\dot{O}$.

For a later objective it is necessary, as the metric ds^2 of our problem from (46) varies for $0 < y_n < \gamma$, that a given surface $y_n = \delta\,(< \gamma)$ be geodesically concave, in other words geodesically convex in the direction of decreasing y_n. This is possible, if instead of ds^2 one uses the metric

$$d\tilde{s}^2 = \lambda\,(y_n)\, ds^2,$$

where

$$\lambda\,(y_n) = \begin{cases} 1 & \text{for } y_n \geq \gamma \\ \left(\frac{\gamma - y_n}{\gamma - \delta}\right)^k + 1 & \text{for } 0 < y_n \leq \gamma \end{cases} \tag{62}$$

and k is a sufficiently large natural number.

We have

$$
\begin{aligned}
\lambda(\gamma) &= 1, \quad \lambda(\delta) = 2 \\
\lambda'(\gamma) &= \lambda''(\gamma) = \cdots = \lambda^{(k-1)}(\gamma) = 0, \quad \lambda'(\delta) = -k\,.
\end{aligned}
$$

Note that $\lambda(y_n)$, and therewith the coefficients of the quadratic form $d\tilde{s}^2$, are obviously $(k-1)$-times continuously differentiable. The surface $y_n = \delta$ is geodesically concave in the direction of increasing y_n, provided that k is sufficiently large. Then the quadratic form (61) for the metric $d\tilde{s}^2$ reads

$$
\begin{aligned}
\sum_{r,s=1}^{n-1} \frac{\partial \tilde{g}_{rs}}{\partial y_n} y'_r y'_s &= \sum_{r,s=1}^{n-1} \frac{\partial \lambda(y_n)\, g_{rs}}{\partial y_n} y'_r y'_s \\
&= \lambda(y_n) \sum_{r,s=1}^{n-1} \frac{\partial g_{rs}}{\partial y_n} y'_r y'_s + \frac{\partial \lambda(y_n)}{\partial y_n} \sum_{r,s=1}^{n-1} g_{rs} y'_r y'_s\,. \qquad (63)
\end{aligned}
$$

Herein the last sum is a positive definite quadratic form, because

$$
ds^2 = \sum_{r,s=1}^{n-1} g_{rs} y'_r y'_s + g_{nn}\, {y'_n}^2
$$

is positive definite. Therefore, for sufficiently large k, (63) is negative definite on the spot $y_n = \delta$, because $\lambda(\delta) = 2$ and $\lambda'(\delta) = -k$. Thus the surface $y_n = \delta$ is in fact geodesically concave toward the side of increasing y_n.

§ 7.
Proof of Existence.

We shall use the following facts from differential geometry. Let $\mathfrak{M}^n$ be an n-dimensional Riemannian manifold with three-times continuously differentiable fundamental values $\dot{g}_{\rho\sigma}$, and let $\overline{\mathfrak{R}}$ be a compact subset of $\mathfrak{M}^n$. Then there is a number $d > 0$ with the properties[5]:

1. Two points P and Q of $\overline{\mathfrak{R}}$, whose distance[6] $\rho(P, Q)$ is $\leq d$, can be joined by a geodesic curve[7], which is determined by being shorter than all other piecewise smooth curves connecting P and Q. We call such curves connecting P with Q *elementary curves*, and d is an *elementary length* that pertains to $\overline{\mathfrak{R}}$.

[5]The proof can be found in SEIFERT and THRELFALL, Variationsrechnung im Großen (Leipzig 1938, Hamburger Einzelschrift 24), p. 97 Note 20. It is shown there for a closed Riemannian manifold $\overline{\mathfrak{R}} = \mathfrak{M}^n$. But there the closure is used only in the application of the Heine-Borel Covering Therem, so the unchanged proof is valid for a compact subset $\overline{\mathfrak{R}}$ of an arbitrary Riemannian manifold.

[6]The distance between P and Q is understood to be the greatest lower bound of the length of all piecewise smooth curves connecting P and Q.

[7]If $P = Q$, the elementary curve consists of only a single point.

2. Let the elementary curve PQ be parameterized by σ, varying from 0 to 1, proportional to arc length. Then the point at the parameter value σ is continuously dependent on σ, P, and Q.

A polygon, whose vertices lie in $\overline{\mathfrak{R}}$ and whose edges are elementary curves, we call an *elementary polygon.*

We now apply these concepts to our orbit problem. The points of $\overline{\mathfrak{E}}$, for which $y_n < \alpha$ holds, must form a normal neighborhood of the boundary $\overline{\mathfrak{E}} - \mathfrak{E}$, that is chosen so small, that all surfaces $y_n = c$, with $0 < c \leq \alpha$, are geodesically convex in the direction of increasing y_n. Let $\overline{\mathfrak{U}}$ be the set of those points of $\mathfrak{E}$ that do not belong to this neighborhood. Just as $\overline{\mathfrak{U}}$ is defined with the help of α, we define three further (closed) regions $\overline{\mathfrak{B}}$, $\overline{\mathfrak{C}}$, $\overline{\mathfrak{D}}$ with the help of the numbers β, γ, δ, where $\alpha > \beta > \gamma > \delta > 0$. All of these regions are homeomorphic to the n-dimensional ball, since one can contract $\overline{\mathfrak{E}}$ onto them along the orthogonal trajectories of the surfaces $y_n = \text{const.}$ Thus it makes sense to speak of diameters in $\overline{\mathfrak{U}}$, which are the images of Euclidean diameters of an archetypal n-dimensional Euclidean ball, mapped topologically onto $\overline{\mathfrak{U}}$ (Fig. 4). We denote the sets of the interior points of $\overline{\mathfrak{U}}$, $\overline{\mathfrak{B}}$, $\overline{\mathfrak{C}}$, $\overline{\mathfrak{D}}$ by $\mathfrak{U}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$.

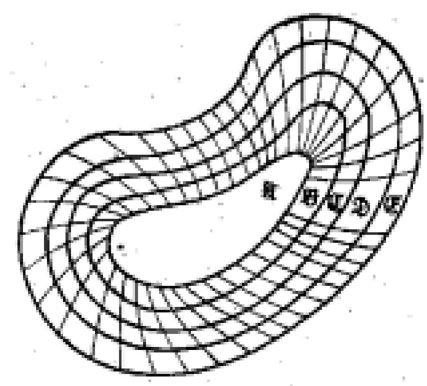

Fig. 4.

Now we choose an elementary length for $\overline{\mathfrak{U}}$, and we make it small enough that each elementary curve whose end points lie in $\overline{\mathfrak{U}}$, lies entirely in $\overline{\mathfrak{B}}$. We consider the collection of the diameters of $\overline{\mathfrak{U}}$ and partition each diameter into N parts so that the distance between successive partition points is no more than the elementary length and so that the partition points are continuous with those on the other diameters. This is possible according to the theorem on uniform continuity, since one needs only to partition the diameters of the archetypal Euclidean ball into N equal Euclidean parts.

Henceforth we replace the diameters of $\overline{\mathfrak{U}}$ by the elementary polygons that one obtains when one connects successive partition points by elementary curves, and completes these elementary polygons on both of their end points by the "radial line segments" $y_1 = \text{const.}, y_2 = \text{const.}, \ldots, y_{n-1} = \text{const.}$ of the spherical shell $\overline{\mathfrak{B}} - \mathfrak{U}$. In this fashion we obtain a family of curves $\mathfrak{S}_B$ of $\overline{\mathfrak{B}}$. Let B be the maximum length of these curves. — A family of curves $\mathfrak{S}_D$ of $\overline{\mathfrak{D}}$ is produced from $\mathfrak{S}_B$, when we attach the radial line segments of the spherical shell $\overline{\mathfrak{D}} - \mathfrak{B}$ to the curves of $\mathfrak{S}_B$. These curves are all shorter than $B + 2b$, where b signifies the length of the radial line segments of the region $\overline{\mathfrak{E}} - \mathfrak{B}$.

Hereafter we change the Riemannian metric ds^2 in the spherical shell $\mathfrak{E} - \mathfrak{C}$ to the metric

$$d\tilde{s}^2 = \lambda\left(y_n\right) ds^2$$

that is given through (62). We choose k therein large enough that the boundary of $\mathfrak{D}$ becomes geodesically convex. We call a boundary surface of a region convex, when it is convex in the sense of the direction from inside toward outside; thus in the case under discussion, in the sense of decreasing y_n. Both of the boundaries of the spherical shell $\overline{\mathfrak{D}} - \mathfrak{B}$ are then both geodesically convex. — The curves of the family $\mathfrak{S}_D$ are shorter than $B + 4b$ in the metric $d\tilde{s}$, since $\lambda \leq 2$ on $\overline{\mathfrak{D}} - \mathfrak{C}$. With respect to the metric $d\tilde{s}$, in $\overline{\mathfrak{D}}$ we choose an elementary length $\tilde{d}$, that is small enough so that the elementary curves, whose endpoints lie in $\overline{\mathfrak{D}} - \mathfrak{B}$, themselves belong to $\overline{\mathfrak{D}} - \mathfrak{B}$. This is possible because of the geodesic convexity of $\overline{\mathfrak{D}} - \mathfrak{B}$. That is to say, if there were no such $\tilde{d}$, then there would be arbitrarily short geodesic curves, whose end points lie in $\overline{\mathfrak{D}} - \mathfrak{B}$, but which jut out of $\overline{\mathfrak{D}} - \mathfrak{B}$. Then they touch the surfaces $y_n = \text{const.}$, the outsides of $\overline{\mathfrak{D}} - \mathfrak{B}$, but lie in arbitrary neighborhoods of $\overline{\mathfrak{D}} - \mathfrak{B}$. That contradicts the geodesic convexity of these surfaces.

On the curves of $\mathfrak{S}_D$ we turn now to taking the metric $d\tilde{s}$ as the basis for a well-known contraction procedure[8], that consists of two steps:

Step 1: Each curve v of $\mathfrak{S}_D$ is divided into q equal length curves, where q is taken large enough that the subcurves are shorter than the elementary length $\tilde{d}$. Each of the curves is replaced by its chord; that is, the elementary curves connecting their end points.

Step 2: In the thus-resulting q-sided elementary polygons the midpoints of the sides are marked and are connected through a new elementary polygon, that is completed as an elementary polygon $f\left(v\right)$ of $q+1$ sides by running the midpoints of the first and last sides to the same boundary points as v.

Note that $f\left(v\right)$ is shorter than v, except when v is a geodesic.

On $f\left(v\right)$ one can apply the same procedure again and thus obtain an elementary polygon $f^2\left(v\right)$. One can continue in this manner without limit.

The families of curves $f\left(\mathfrak{S}_D\right), f^2\left(\mathfrak{S}_D\right), \ldots$, that are generated thus from $\mathfrak{S}_D$, all have the property of covering $\overline{\mathfrak{D}}$ completely, as will be proved in § 8. Hence, in the ν^{th} family of curves there are certain curves, whose intersection with $\mathfrak{B}$ is not empty. Let the curve v_ν be

[8] Worked out in more detail in Seifert and Threlfall, Variationsrechnung im Großen (Leipzig 1938).

The fundamental idea of the existence proof, namely through tightening one of the manifold's covering family of curves to construct the desired geodesic, is familiar from G. D. Birkhoff, Dynamical Systems (New York 1927), where it is used in the construction of a closed geodesic on a convex surface.

Whether the existence of n geodesics going from boundary to boundary can be proved with the method of von Lusternik and Schnirelmann, I can not say, because the pertinent literature is not at my disposal.

one such curve v, that still touches $\mathfrak{B}$ after application of the contraction procedure ν times. Then we have the formulas

$$(64) \qquad f^{\nu}(v_{\nu}) \cdot \mathfrak{B} \neq 0; \qquad (\nu = 1, 2, 3, \ldots)$$

where the point $\cdot$ denotes intersection of sets.

But we also have

$$(65) \qquad f^{\nu}(v_{\nu+m}) \cdot \mathfrak{B} \neq 0\,. \qquad (m = 1, 2, \ldots)$$

That is, if this inequality were false, then $f^{\nu}(v_{\nu+m})$ would lie in $\overline{\mathfrak{D}} - \mathfrak{B}$, and would stay in $\overline{\mathfrak{D}} - \mathfrak{B}$ under incessant application of f, since the boundary of $\mathfrak{D} - \mathfrak{B}$ is geodesically convex. Thus $f^{\nu+m}(v_{\nu+m})$ would be disjoint to $\mathfrak{B}$, which contradicts (64).

From the sequence $v_1, v_2, \ldots$ we now choose a subsequence converging to a certain polygon v. To achieve this, we require only making a selection such that we have convergence of the end points of the curves of the subsequence (situated diametrically opposite, on the boundary of $\mathfrak{D}$). We denote by $v_{\nu_1}, v_{\nu_2}, \ldots$ the subsequence thus obtained. Then for all ν we have

$$(66) \qquad f^{\nu}(v) \cdot \bar{\mathfrak{B}} \neq 0\,.$$

That is, if this inequality were false for a particular ν, then $f^{\nu}(v)$ would lie in $\overline{\mathfrak{D}} - \overline{\mathfrak{B}}$. Then all but a sufficiently adjacent approximating elementary polygon would also lie in $\overline{\mathfrak{D}} - \overline{\mathfrak{B}}$. Thus we would have

$$f^{\nu}(v_{\nu_i}) \cdot \bar{\mathfrak{B}} = 0 \text{ for suitable } \nu_i > \nu,$$

in contradiction to (65).

From the sequence of the $(q+1)$-sided elementary polygons $f(v), f^2(v), \ldots$ we choose a convergent subsequence through iterated selection, in which we first ensure that the sequence of the first vertices converges, then through further selection the sequence of the second vertices is made to converge, and so on. We denote the limit polygon by w. Since $\bar{\mathfrak{B}}$ is closed, it follows from (66) that

$$(67) \qquad w \cdot \bar{\mathfrak{B}} \neq 0\,.$$

Note that w is a $(q+1)$-sided elementary polygon without vertices, thus a geodesic. That is, if w were to have a vertex, then $f(w)$ would be shorter than w. Hence $f^{\nu}(v)$, for sufficiently large ν, would also be shorter than w. That would stand in contradiction to the fact that $f^{\nu}(v)$ converges to w as $\nu \to \infty$, and therewith the length of $f^{\nu}(v)$ converges monotonically downward to the length of w.

We consider a point on w belonging to $\overline{\mathfrak{B}}$, and from it we follow w outward toward both sides as far as the first intersection points with the boundary of $\mathfrak{C}$. The fragment thus obtained we call g. It is geodesic in the metric ds and shorter than $B + 4b$; because in $\mathfrak{C}$ the metrics ds and $d\tilde{s}$ are in agreement.

Now let γ_ν and δ_ν be greater than zero and $\delta_\nu < \gamma_\nu$. From the numbers γ_ν, δ_ν we construct the regions $\mathfrak{C}_\nu$ and $\mathfrak{D}_\nu$, just as $\mathfrak{C}$ and $\mathfrak{D}$ were constructed previously from γ and δ, and a geodesic curve g_ν, that goes through $\bar{\mathfrak{B}}$, shorter than $B + 4b$ and whose end points lie in $\overline{\mathfrak{C}}_\nu - \mathfrak{C}_\nu$. We consider a point R_ν of g_ν that belongs to $\bar{\mathfrak{B}}$, and the tangent vector e_ν to g_ν at that point. From the sequence $e_1, e_2, \ldots$ we choose a convergent subsequence. Let e^* be the limit vector and R^* be its base point. R^* and e^* determine a geodesic g^*. On neither side of R^* can one traverse the length $B + 4b$ along g^*. That is, suppose that this were possible toward one side. Then the thus traversed, closed geodesic curves of the length $B+4b$ would lie in a certain $\mathfrak{C}_\nu$; therefore also the geodesic curves of the length $B+4b$, that are determined through the neighboring vectors e_μ with sufficiently large μ. Hence one end point of g_μ would lie in $\mathfrak{C}_\nu$. But that is not the case for $\mu > \nu$.

The lengths, that one can traverse along g^* from R^* out toward one side or the other, thus have the property that they are finite with the upper limits l and r. Hence g^* is a geodesic of finite length $l + r$, that cannot be extended. Therefore, as we stated at the end of § 5, g^* goes from boundary to boundary and pertains to a periodic motion of the mechanical system.

§ 8.
Lemma on Families of Curves.

We must yet convey the proof that the families of curves, that arise out of $\mathfrak{S}_D$ through repeated application of the contraction procedure, cover $\overline{\mathfrak{D}}$ completely.

The families of curves $f^\nu(\mathfrak{S}_D)$ all have the following properties: They consist of $(q+1)$-sided elementary polygons, the diametrically opposite points belong to the boundary of $\overline{\mathfrak{D}}$, and their vertices depend continuously on the end points lying on the boundary $\overline{\mathfrak{D}} - \mathfrak{D}$.

In particular: If P, P' is a pair of diametrically opposite points in $\overline{\mathfrak{D}} - \mathfrak{D}$, and we denote by $(PP') = (P'P)$ the elementary polygon that connects P with P' in the family $f^\nu(\mathfrak{S}_D)$, we denote its vertices by

$$P = P_0, P_1, \ldots, P_q, P_{q+1} = P' \tag{68}$$

or also, going out from the other vertex, by

$$P' = P'_0, P'_1, \ldots, P'_q, P'_{q+1} = P,$$

then for a fixed k, the vertex P_k depends continuously on the point P. But from this property alone it follows that the family of curves covers $\overline{\mathfrak{D}}$.

For the proof we consider an n-dimensional projective space, endowed with the elliptical metric. We think of it as a Euclidean space that is closed through the infinitely distant hyperplane. We subtract the closed unit ball from it and call the remaining space $\mathfrak{B}$ and its closed hull $\overline{\mathfrak{B}}$. Then the family of curves $f^\nu\,(\mathfrak{S}_D)$ can be associated with a continuous mapping from $\overline{\mathfrak{B}}$ to $\overline{\mathfrak{D}}$, in the following manner:

To begin with, we map the boundary sphere $\overline{\mathfrak{B}} - \mathfrak{B}$ topologically onto the boundary sphere $\overline{\mathfrak{D}} - \mathfrak{D}$ so that diametrically opposite points are mapped into diametrically opposite points. When the diametrically opposite points P, P' from $\overline{\mathfrak{D}} - \mathfrak{D}$ meet the diametrically opposite points Q, Q' from $\overline{\mathfrak{B}} - \mathfrak{B}$, then we map the projective line segment QQ' that lies in $\overline{\mathfrak{B}}$ onto the elementary polygon PP', so that a sequence of elliptically equidistant points

$$Q = Q_0, Q_1, \ldots, Q_q, Q_{q+1} = Q'$$

is mapped to the sequence (68) of the vertices of the elementary polygon, and the subproportions of all the subparts remains preserved.

This mapping is continuous, because according to property 2 on page 18, the point R of $\overline{\mathfrak{D}}$ that divides one fixed elementary line segment P_kP_{k+1} in the proportion $\sigma : (1-\sigma)$, depends continuously on σ, P_k, and P_{k+1} and therefore continuously on the boundary point P and σ, since P_k and P_{k+1} depend continuously on P. Finally, P depends continuously on Q, thus it follows that R depends continuously on Q and σ.

Now each continuous mapping from $\overline{\mathfrak{B}}$ to $\overline{\mathfrak{D}}$, that maps the boundary sphere $\overline{\mathfrak{B}} - \mathfrak{B}$ topologically onto the boundary sphere $\overline{\mathfrak{D}} - \mathfrak{D}$, is a mapping of degree 1 (mod 2). (We must use the mapping degree mod 2 because $\overline{\mathfrak{B}}$ is not orientable for even dimensions n.) Then each continuous mapping B from $\overline{\mathfrak{B}}$ to $\overline{\mathfrak{D}}$, that maps the boundary spheres in the indicated manner, can be deformed into the mapping A, that maps the projective lines that pass through the origin, hence because they lie in $\overline{\mathfrak{B}}$, into the diameters of $\overline{\mathfrak{D}}$. In order to deform the mapping B into the mapping A, one has only to connect the image point of an original point of $\overline{\mathfrak{B}}$ under the mapping B with its image point under A in $\overline{\mathfrak{D}}$ through a "rectilinear extension" — rectilinear understood in the original Euclidean sense, with the help of which the diameters of $\overline{\mathfrak{D}}$ were also defined — and, in the rectilinear connecting extensions, the first image point is allowed to travel to the second image point with constant speed in the unit time interval. Clearly the mapping A is a mapping of degree 1, since it maps the entirety of $\overline{\mathfrak{B}}$ topologically onto $\overline{\mathfrak{D}}$ with the exception of the infinitely distant hyperplane of $\overline{\mathfrak{B}}$, which is mapped into the center point of $\overline{\mathfrak{D}}$. Since the mapping degree does not change under deformation, it follows that the aforementioned mapping B is also a mapping of degree 1, and therefore each point of $\overline{\mathfrak{D}}$ is the image of at least one point of $\overline{\mathfrak{B}}$; that is, at least one curve of the family $f^\nu\,(\mathfrak{S}_D)$ passes through each point of $\overline{\mathfrak{D}}$.

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