INTRODUCTION TO A PAPER OF M.Z. SHAPIRO: HOMOTOPY THEORY IN CONTROL

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1 The Results

A curve is called regular if its derivative is never zero. We associate to any regular curve $\mathbf{x}(t)$ on the unit two-dimensional sphere a moving frame $f(t) = [\mathbf{x}(t), \mathbf{T}(t), \mathbf{N}(t)]^t$ whose row vectors are $x(t)$, its unit tangent vector $\mathbf{T}(t) = \dot{\mathbf{x}}(t)/|\dot{\mathbf{x}}(t)|$ and its righthanded normal $\mathbf{N}(t) = \mathbf{x}(t) \times \mathbf{T}(t)$. Thus $f(t)$ is a curve in the three-dimensional rotation group $S0(3)$. It satisfies the Frenet–Serret equations

$$\frac{d}{dt} f = \begin{pmatrix} 0 & v & 0 \\ -v & 0 & k \\ 0 & -k & 0 \end{pmatrix} f \tag{1}$$

(These are the standard Frenet-Serret equations for the space curve $\int_0^t x(s)ds$.) Here v is the speed of the curve $x(t)$ and k/v is its curvature. (1) defines a right-invariant distribution of two-planes on $SO(3)$. (It is the distribution mentioned by John Baillieul at the beginning of his talk.) It defines a control system with controls v, k. We must impose the bound $v > 0$ since we are interested in regular curves.

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Fix an initial frame and a final frame. Consider the space of all solutions $f(t)$ to the Frenet-Serret control system (1) which have these as their initial and final frames.

Question: How many connected components does this space have?

Theorem 1 (Smale, 1958) *Two, as indicated:*

Now suppose instead we also impose the constraint $k > 0$. The resulting class of curves on the two-sphere are called right-handed nondegenerate. (Their curvature is always positive.) And let us ask the same question.

Theorem 2 (J. Little, 1970) *There are* **three** *components of the space of solutions to the control system with the constraints $v > 0$, $k > 0$ imposed, provided the the inital and final frames are equal. The representatives of these components are indicated:*

B.Z. and M.Z. Shapiro [BZMZ] have shown that the difference $1 = 3 - 2$ between Little's and Smale's theorem is a consequence of whether or not curves cross the boundary of the small-time accessible set. Our goal is to popularize their results and suggest that questions of homotopy theory may be important to control theory.

M.Z. Shapiro investigated the following generalization of Little's problem to n-dimensions:

$$f(t) \in SO(n),$$

the n-dimensional rotation group.

$$\frac{df}{dt} = \begin{pmatrix} 0 & k_1 & 0 & & & & \\ -k_1 & 0 & k_2 & & & & 0 \\ 0 & -k_2 & 0 & \ddots & & & \\ & \ddots & & \ddots & \ddots & & \\ & & \ddots & & \ddots & \ddots & \\ & 0 & & \ddots & & \ddots & k_{n-1} \\ & & & & -k_{n-1} & & 0 \end{pmatrix} f$$

$$k_i > 0, i = 1, 2, \ldots, n-1$$

We call this the Frenet-Serret distribution, or the Cartan distribution. It is a right-invariant distribution of cones on $S0(n)$.

Theorem 3 (M.Z. Shapiro, 1992) *The space of solution curves to the above control system on $SO(n)$ which connect a frame $f_0 \in SO(n)$ to itself has exactly two components if n is even and exactly three components when n is odd.*

2 Why should we care?

1) These theorems count the number of connected components of solutions to control problems with fixed endpoints. Two curves lie in the same component if and only if it is possible to find a one-parameter family of control strategies, $u_s(t)$, $0 \le s, t \le 1$ (and so a two-parameter family of controls) such that $u_0(t)$ leads to the first curve, $u_1(t)$ to the second, and all of the intermediate curves have the same endpoints. To put it more briefly, the first curve can be deformed into the second by a control-induced homotopy which fixes the endpoints.

For example, consider a man with two different control strategies which lead to the same position of his hand gripping a bar.

You cannot homotope from one to the other without breaking contact. Try it!

2) There are typically an uncountable number of solutions to the problem of finding controls connecting two given points. Counting the connected components of this solution space provides a meaningful way to count distinct solutions.

3) One of the main tools used in proving the theorems mentioned is the "covering homotopy property". This is really already part of a control theorists foolbox. (See eg. Sussmann's talk in this proceedings.) A topological

perspective should provide insight into the use and importance of this tool in control.

3 Why Two?

We begin by recalling some basic notions from homotopy theory. The set of path-components of a space X is denoted by $\pi_0(X)$. The space of closed continuous loops of a connected space Q based at $q_0 \in Q$ ($\gamma(0) = \gamma(1)) = q_0$) is denoted $\Omega(Q)$ or sometimes $\Omega(Q, q_0)$. The fundamental group of Q is

$$\pi_1(Q) = \pi_0(\Omega(Q)).$$

Its elements are called homotopy classes (of based loops) and it forms a group.

It is well known that $\pi_1(S0(3))$ is the two-element group so that

$$\#\pi_0(\Omega(S0(\xi)) = 2.$$

This is the "2" in the theorem of Smale.

Remark. The identity element $e \in \pi_1(S0(3))$ is represented by the constant path $f(t) \equiv f_0$. The nontrivial element $\sigma \in \pi_1(S0(3))$ is represented by rotation through 2π radians about any axis of space.

Let $\Omega = \Omega(S0(3))$, and let $\Omega_K \subset \Omega$ denote the loop space of Smale's theorem. The answer "2" is a corollary of a deeper result of Smale which states that the inclusion of Ω_K in Ω induces an isomorphism on π_0:

$$i_* : \pi_0(\Omega_K) \simeq \pi_0(\Omega)$$

Here i_* denotes the map which assigns to each connected component of Ω_K the corresponding connected component of Ω which contains it.

This is a surprising result, for given a pair of topological spaces $A \subset B$ there is no reason for the corresponding i_* to be 1-to-1 or onto:

(The black blobs represents $A = \Omega_K$ and the blobs encircling them $B = \Omega$.)

3 Covering homotopies

The notion of a covering homotopy is central to the proofs of the theorems above. It appears naturally in control theory.

Let $p : \mathcal{S} \to Q$ be a continuous map between connected spaces. We have in mind the endpoint map which assigns to each controlled path beginning at q_0 its endpoint. In other words, for each control strategy $u(\cdot)$, solve the control system $\dot{q} = f(q(t), u(t))$, with initial condition $q(0) = q_0$. Then $p(u(\cdot)) = q(1)$.

Definition 1 *We say that p satisfies the 1-parameter covering homotopy property, or CHP for short, if for each path $q(s)$, $1 \leq s \leq 2$ in Q and any $\gamma_1 \in \mathcal{S}$ with $p(\gamma_1) = q(1)$ there exists a path $\gamma(s)$, $1 \leq s \leq 2$ covering $q(s)$: $p(\gamma(s)) = q(s)$.*

In other words, the 1-parameter CHP holds if we can follow any motion $q(s)$ of final states by an appropriate two-parameter families of controls $u(t, s)$.

The salient result from homotopy theory is that if $\mathcal{S}$ is contractible, and p satisfies the 1-parameter CHP then

$$\pi_0(p^{-1}(q_0)) = \pi_i(Q).$$

(This follows immediately from the exact homotopy sequence.) Since $\pi_1(Q) = \pi_0(\Omega(Q))$ this in turn implies that $\#(\pi_0(\Omega(Q)) = \#\pi_0(\Omega_K)$ as in Smale's theorem.

It follows from Little's theorem that the 1-parameter CHP must fail for his system. Let us see **how** it fails. Consider the following set-up for testing the CHP:

Here we are to swing the final frame f_1 of the initial nondegenerate curve $\gamma_1(t)$ across the equator defined by $\dot{\gamma}_0(0)$. This equator is indicated by the vertical dashed curve.

Now consider the central projection of this figure on to a tangent plane. (By a central projection we mean a stereographic projection with light source at the sphere's center.)

Central projection preserves nondegeneracy of curves. Now any planar curve with initial frame f_0, and final frame f_2, and *no self-intersections* must have *an inflection point*. See the following figure.

(Cf. Arnol'd, [1].) At inflection points the curvature is zero and so the control bound $k > 0$ is violated. Thus any homotopy γ_s which follows the frames f_s must leave the space Ω_K of solution curves. This shows that the endpoint map for Little's distribution violates the 1-parameter CHP.

Remark The reason behind considering central projections comes from projective geometry. The sphere is the universal cover of the projective plane. The central projections then become the standard affine charts of projective geometry. The Frenet-Serret distribution is perhaps most properly thought of as having to do with projective geometry. In particular, as shown in the last section, it induces a distribution on projective frames and is invariant under projective transformations.

If instead, the initial choice γ_1 has a self-intersection then it becomes possible to cover the curve f_s with a homotopy γ_s of γ_1. This is indicated in the picture below.

The central step in Shapiro's proof is isolating that subset of the space of nondegenerate curves for which the 1-parameter CHP fails. These are the disconjugate curves described which we now.

4 Conjugate and Disconjugate Curves.

Definition 2 *A nondegenerate curve $x(t)$ in S^2 is called disconjugate if it intersects any great circle no more than 2 times. It is called strictly conjugate if it intersects some great circle three times transversely.*

For the reason behind this terminology see the remark in the next subsection.

The theorem of Shapiro-Little follows directly from the following:

Theorem 4 *The set of disconjugate loops and the set of conjugate loops are disconnected within the space of all nondegenerate loops. The disconjugate loops form a contractible set within the set of all nondegenerate loops. The 1-parameter CHP holds for the space of nondegenerate curves minus the disconjugate curves.*

5 Higher-dimensional homotopies and other generalities

The 1-parameters CHP requires us to follow a 1-parameter family of targets by an appropriate family of control strategies. Suppose instead that we want to follow k-parameter target sets. Let Σ^j denote the j-dimensional cube and I the unit interval.

Definition 3 *The k-parameter CHP holds for $p : \mathcal{P} \to Q$ provided whenever $f : \Sigma^{k-1} \times I \to Q$ is a continuous map and $\gamma : \sum^{k-1} \to \mathcal{P}$ is another continuous map such that $\pi(\gamma(\sigma)) = f(\sigma, 0)$ then there is a map $\Gamma : \sum^{k-1} \times I \to \mathcal{P}$ with $\Gamma(\cdot, 0) = \gamma(\cdot)$ and $p \circ \Gamma = f$.*

If the k-parameter CHP holds for all k we will say, that p satisfies the CHP.

Let us return to Smale's paper. Let Q be a connected Riemannian manifold. Fix a point q_0 in Q and a unit vector v_0 attached there. Recall a curve is called regular if its derivative is nowhere zero. Let $\mathcal{P}^{\text{reg}}$ denote the space of all continuously differentiable regular paths in Q beginning at a point q_0 and with initial direction v_0. Let STQ denote the space of all unit tangent vectors. Consider the map $p : \mathcal{P}^{\text{reg}} \to STQ$ which assigns to each path the value $(\gamma(1), \dot{\gamma}(1) \, // \|\dot{\gamma}(1)\|)$ of its final tangent direction.

[Smale]
Theorem 5 *This map p satisfies the CHP.*

Now it is a general fact (again following from the exact homotopy sequence) that if $p : \mathcal{P} \to Y$ satisfies the CHP, if $\mathcal{P}$ is contractible, and if Y is connected then the 'fibers' $p^{-1}(y)$ and the space of loops $\Omega(Y)$ on Y are *weakly homotopy equivalent*. To say that two spaces are weakly homotopy equivalent means that all their homotopy groups agree:

$$\pi_k(p^{-1}(y)) \simeq \pi_k(\Omega(Y)) = \pi_{k+1}(Y).$$

(If Ω is a topological space, then $\pi_k(\Omega)$ is the space of path-connected components of the space of maps of a k-sphere into Ω.)

Smale showed that $\mathcal{P}^{\text{reg}}$ was contractible. Since $STS^2 = S0(3)$ his theorem stated at the beginning follows immediately from this one.

Consider the particular case when Q is a two-dimensional surface. Let $e_1(q)$, $e_2(q)$ be a local orthonormal frame on Q so that any unit vector u can be written $\vec{u} = \cos\varphi e_1 + \sin\varphi e_2$. Then $\dot{q} = v\vec{u}, v > 0$ is our control law. It can be rewritten as the Pfaffian system

$$\omega \equiv -\sin\varphi\theta_1 + \cos\varphi\theta_2 = 0$$

where θ_1, θ_2 are the dual basis to e_1, e_2.

This system is of contact type: $w \wedge dw = 0$. As a baby version of his main theorem, Smale proves the following

Theorem 6 *Let D be a contact distribution on a connected 3-manifold. Let Ω_D denote the set of all absolutely continuous Legendrian ($\dot{\gamma} \in D$) loops through a fixed point. Then the inclusion of this space into the space of all loops is a weak homotopy equivalence.*

In this same vein, Ge Zhong and independently Sarychev have proved the following.

Theorem 7 *Let D be a bracket generating distribution on a connected manifold Q. Then the inclusion $\Omega_D \hookrightarrow \Omega$ of the horizontal ($\dot{\gamma} \in D$) absolutely continuous loops through a fixed point into the space of all loops through that point induces a weak homotopy equivalence.*

Their proofs follow the main lines of Smale's. All additional difficulties are taken care of by invoking Chow's theorem, as the reader may have guessed.

As we can see from the results of Little and Shapiro, the situation becomes much more interesting when we impose inequality constraints on the controls. In fact, the situation become more interesting if we simply impose more smoothness on our controls. This is evidenced by the existence of C^1-rigid curves as defined by Bryant-Hsu. (The simplest example of such a curve is any segment of the x-axis for the control system $dz - y^2 dx = 0$ on $\mathbf{R}^3$.)

6 Problems

In order to organize our thoughts we will now state some general problems.

Suppose we are given a distribution $k \subset D \subset TQ$ of cones where Q is a smooth connected manifold, D a bracket generating distribution and $k_q, q \in Q$ a family of cones varying smoothly with q. Fix two points q_0, q_1 and let $\Omega_K^r = \Omega_K^r(q_0, q_1)$ denote the set of all r-times continuously differentiable paths γ joining q_0 to q_1 and satisfying the control system $\dot{\gamma} \in K_\gamma$. When $r = 0$ take the paths to be absolutely continuous and write $\Omega_K^0 = \Omega_K$, $\Omega_{TQ}^r = \Omega^r$.

Problem 1: How many path-connected components does Ω_K^r have?

Problem 2: Let Ω denote the space of *all* paths joining q_0 to q_1 (no conditions on controls as smoothness). Is $\Omega_K^r \hookrightarrow \Omega$ a weak homotopy equivalence?

Problem 3: Does the answer to problem 1 depend on the degree of smoothness r?

Problem 4: How does the answer to problem 1 vary as we vary the end points?

Problem 5: How does the answer to Problem 1 vary as we vary the opening angle of the cone? If the original cone is open in D, can the answer change if we take its closure?

Regarding problems 1,3,4. In the case of the stable abnormal simple curve the in [3] mentioned above, when $q_0 \neq q_1$ are two points on this curve we have

$$\#\pi_0(\Omega_D^0) = 1$$
$$\#\pi_0(\Omega_D^1) = 2$$

But if q_1 is not on the curve then

$$\#\pi_0(\Omega_0^r) = 1, \quad r = 0, 1, 2, 3, \ldots$$

Such phenomena is impossible when there are no controls ($D = TQ$):

$$\pi_0(\Omega^r) = \pi_0(\Omega).$$

Regarding problem 5. The results of Smale-Little-Shapiro show clearly that indeed the answer can depend on the cone's opening angle.

Regarding problem 2. *Relaxing controls and the h-principle of Gromov*. To say that the answer to this problem is 'yes' means that by completely relaxing the controls, then we get the "right answer" for the topology. In this situation we say that Gromov's weak h-principle applies. His regular *h-principle* is that

$$i_* : \pi_o(\Omega_k) \to \pi_0(\Omega)$$

is onto. This means every homotopy class of path is represented by a control path. His 1-*parameter h-principle* states that i_* is 1-to-1. This means that if two control paths (with fixed end points) are homotopic, disregarding the controls, then that homotopy can be realized through a non-parameter family of control strategies (all having the same end points).

(He also has C^r-versions of his principle.)

Thus we can summarize: In Smale's case the weak h-principle applies. The 1-parameter h-principle fails for the examples of Shapiro and Little.

Warning: To say that the weak h-principle holds is stronger than saying that the regular or 1-parameter h-principle. This is an accident of historical nomenclature we are stuck with.

7 Disconjugacy in Higher Dimensions

It is not immediately clear how to generalize the Little result to higher dimensions. To do this the viewpoint of projective geometry appears essential.

Call a curve $\gamma(t)$ in ^{n} vector nondegenerate, or VN for short, if $\gamma(t)$, $\dot\gamma(t)$, $\ddot\gamma(t), \ldots, \gamma^{(n-1)}(t)$ are linearly independent vectors in ^{n} for each t. Call it right-handed or (RHVN for short) if in addition this basis is positively oriented.

The Graham-Schmidt procedure allows us to pass from RHVN curves to a curves in $S0(n)$ satisfying the Frenet-Serret equations $(*)$. We can also associate to a RHVN curve a moving family of subspaces of ^{n}. Namely, set $f_1(t) = \text{span}\ \{\dot\gamma(t)\}$ and let $f_j(t)$ denote the linear span of the first j

derivatives of γ. By assumption, $dim(f_j(t)) = j$, and

$$0 \subset f_1(t) \subset f_2(t) \subset \ldots \subset f_j(t) \subset \ldots f_{n-1}(t) \subset {}^n.$$

A collection of such subspaces is called a (complete) flag

Each subspace f_j is *oriented* by the ordering of the derivative vectors $\gamma^{(j)}(t)$. Denote the set of all oriented flags ^{n} by SF_n. $S0(n)$ acts freely and transitely on SF_n, and so this action defines a diffeomorphism $S0(n) \simeq SF(n)$. The Frenet-Serret control system on $SO(n)$ mentioned in the beginning has a beautiful description on SF_n. Let e_1 denote the vector field on SF_n defined by rotating the line f_1 of any given flag within the plane f_2 according to the positive sense of rotation defined by the orientations. And in general let e_i, $i = 1, 2, \ldots, n-1$ be the vector field defined by rotating the i-dimensional subspace $f - i$ about the $(i-1)$st keeping it within the $(i+1)$st, and so that the rotation is in a positive sense. Then our control system is is

$$\dot{f} = \sum_{i=1}^{n-1} k_i e_i(f) \ \ ; k_i > 0.$$

We will call this description the "projective description" of our control system.

From the projective description it becomes obvious that the full linear group, $Gl(n)$, is the symmetry group of our control system. From our original description we could only see that the smaller group $SO(n)$ was a symmetry group. These additional symmetries allow Shapiro to construct explicit covering homotopies. (Any positive scalar multiple of the identity I acts trivially on SF_n, so that the action of $Gl(n)$ actually factors through an action of $GL(n)$ modulo this one-dimensional subgroup. This quotient group is the disconnected double cover of the projective linear group $GL(n) = GL(n)/I$ which is the group of projective transformations.) More importantly, the projective viewpoint allows Shapiro and Shapiro to pinpoints the higher dimensional disconjugate curves, that is, te set of curves on which the 1-parameter CHP fails.

Remark

The motivation for Shapiro to study this system came from a certain Poisson structure called the Gelfand-Dikii structure which arises in the study of completely integrable PDE such as KdV. The underlying manifold for this structure is the affine space of all linear n-th order differential operators:

$$L(y) = y^{(n)} + u_{n-1}(t)y^{(n-1)} + \dots u_1(t)y(t).$$

Let $y_1, \dots y_n$ be any basis for the space of solutions to the equation $L(y) = 0$. Then $(y,(t),\dots,y_n(t))$ is a VN curve in ^{n}. We think of the coefficients u_i as the controls. Let $Y(t)$ be the fundamental matrix solution to such a DE. Thus the ij entry of Y can be taken to be the jth derivation of the solution y_i. We will say that $Y(1)Y(0)^{-1}$ is the monodromy of the nth order differential operator. We will say that two operators are isomonodromic if there monodromy operators are constant. The symplectic leaves of the Gelfand-Dickii Poisson structure consists of the connected components of the isomonodromy classes. Shapiro and Khesin show how to reduce the problem of seperating the connected components to the problem we have been discussing on $SO(n)$ or SF_n.

Exercise. Relate the controls u_i to the Frenet-Serret controls k.

Exercise. Relate this to standard linear control theory, cf. Sontag, p. 133.

Now suppose that $\gamma : I = [0,1] \to {}^n$ is a VN curve, let H be any hyperplane and ℓ a linear function defining H; $H = \{\ell = 0\}$. Define the multiplicity of H at t_0 to be the order of vanishing of $\ell(\gamma(t))$ at t_0. In particular the multiplicity is zero if $\gamma(t_0) \notin H$. The multiplicity is a nonnegative integer less than or equal to $n-1$. It is equal to $n-1$ if and only if H is the osculating plane of γ at t_0. To see this, observe that relative to the basis defined by its derivatives at t_0 we have

$$\gamma(t) = \left(1, (t-t_0), \frac{1}{2}(t-t_0)^2, \dots, \frac{1}{(n-1)!}(t-t_0)^{n-1}\right) + 0(t-t_0)^n).$$

If we perturb H a bit in the correct direction then it will intersect γ transversely in $n-1$ points near p_0 and these points limit to p_0 as H approaches the osculating plane.

Let the multiplicity of γ relative to H be the sum of all nonzero multiplicities.

Definition 4 *γ is a conjugate curve if for all hyperplanes the total multiplicity is at most $n-1$. Otherwise it is disconjugate.*

With this definition in place M.Z. Shapiro proves that the analog of the theorem of Little-Shapiro holds for higher dimensions. A key ingredient in the proof is the notion of the *train* of an initial flat f_0.

Definition 5 *Two flags $f = f_1 \subset f_2 \subset \ldots$ and $e = e_1 \subset e_2 \subset \ldots$ in ^{n} are said to be transverse, or in general position, if the intersections of all their subspaces is as transverse as possible. In other words, they are transverse if for each i, j the dimension of $f_i \cap e_j$ is the minimum possible for such an intersection of subspaces, namely $max(i + j - n, 0)$. The train of the flag f is the set of all flags e which are* **not** *transverse to it.*

Theorem 8 *If the VN γ is not conjugate then its associated flag curve $f(t)$ must intersect the train of its initial flag $f(0)$.*

Remark If we put bounds, eg $\Sigma k_i < 1$, on the controls then near an initial flag f_0 its train is precisely the the boundary of its small-time accessible set.

Remark For those familiar with some elements of Lie group theory, the train is the union of all the lower-dimensional Schubert cells in the cell decomposition of $SO(n)$.

Finally, I should explain to the reader what the difference is between even and odd dimensions. Why do we get 2 when n is even and 3 when n is odd? Because when n is even there are no conjugate loops! The reason for this is simple. The curve must return to its starting place, and hence intersect any hyperplane an even number of times. But the multiplicity of the loops initial osculating plane, and slight perturbations of it, is at least $n - 1$ which is odd. Hence the multiplicity of the curve with respect to such planes at least n and the loop is conjugate.

I hope I have given the reader enough background and motivation to read the papers of Shapiro et al. Bon voyage.

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