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The zero angular momentum, three-body problem: All but one solution has syzygies

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Abstract. A syzygy in the three-body problem is a collinear instant. We prove that, with the exception of Lagrange's solution, every solution to the zero angular momentum, Newtonian three-body problem suffers syzygies. The proof works for all mass ratios.

1. *Introduction*

We consider the Newtonian three-body problem with zero angular momentum and negative energy. Masses are positive, but arbitrary. A 'syzygy' means an instant (or configuration) at which the three masses are collinear.

THEOREM 1.1. *All solutions to the zero angular momentum, negative energy, Newtonian three-body problem admit a syzygy except for the Lagrange homothety solutions.*

1.1. *Explanation of terms.*

Solutions are to be defined over their maximal interval of existence and analytically continued through binary collisions following Levi-Civita [**3,** see especially p. 105, equation (12)]. Binary collisions count as syzygies. Collinear solutions count as being in constant syzygy. A solution cannot be extended past a finite time $t = b$ if and only if as $t \to b$ the three positions of the three bodies tend to the same point. In other words, a solution fails to exist past a certain time if and only if it ends in a triple collision at that time (see [**6, 11**] or [**12**]).

The only explicitly known solutions to the three-body problem are the Lagrange and Euler solutions. At zero angular momentum, these solutions evolve by homothety (scaling). The Euler solutions are collinear at each instant. The Lagrange homothety solution [**2**] for negative energies begins and ends in a triple collision. At every other instant of its existence the three masses form an equilateral triangle. This triangle evolves by homothety, 'exploding' out of the triple collision, growing to its maximal size half-way through its evolution, at which point the three bodies are instantaneously at rest, and then shrinking back to a triple collision. The size of the equilateral triangle at its maximum is determined by the value of the negative energy.

1.2. *Ubiquity of non-Lagrange solutions.* There are many many solutions to the zero angular momentum, negative energy, three-body problem besides those of Euler and Lagrange. The set of all such solutions to this problem forms a continuum which should be viewed as a four-dimensional space, and the Lagrange solution is a *single point* within this continuum. Indeed, the zero angular momentum phase space, modulo the symmetry of rotation, is six-dimensional, being isomorphic to the tangent bundle of a 3-manifold (namely, $\mathbb{R}^3$ minus three rays). This is the reduced phase space for the problem, and Newton's equations define a flow on this space (see, for example, [**5–9**]). Fixing the energy gives us a 5-manifold, say M^5. Formally, then, the space of solutions to which Theorem 1.1 refers is M^5 modulo the $\mathbb{R}$-action defined by the flow of Newton's equations. Hence, in a formal sense the solution space which is the subject of the theorem is a four-dimensional space. (Because of collisions, the $\mathbb{R}$-action is not globally defined. Because of complicated dynamics the quotient space will not be a manifold.) The Lagrange homothety solution at that energy is a single curve in M^5 (see [**6**]). Any initial condition $m \in M^5$ not lying on this curve yields a solution which is not the Lagrange solution and so, according to the theorem, a solution which has syzygies. Thus, the exceptional solution—Lagrange's—is a single point within the four-dimensional space of solutions.

1.3. *Previous work.* In [**7**] Theorem 1.1 was proved upon imposing two additional hypotheses on solutions: that they are bounded and that they do not end in a triple collision. The contribution of the present paper is to dispense with these hypotheses.

We first dispense with the hypothesis on collision, keeping the boundedness hypothesis. Again, in [**7**] it was proved that bounded solutions which do not end in collisions have syzygies. Fujiwara *et al* [**1**] later found another more elementary proof. Either proof, plus invariance of the equations and zero angular momentum condition under time reversal, proves the existence of syzygies for solutions that are bounded and do not begin in a triple collision. All that remains of the bounded solutions are those, excluding the Lagrange solution, which begin and end in a triple collision. The proof for these solutions will follow the same qualitative lines as [**7**]. According to Moeckel [**6,** Corollary, p. 53], there are, for generic mass ratios, an infinite number of these finite-interval solutions bi-asymptotic to a triple collision.

Moeckel, Chenciner and others have pointed out (private communication) that dispensing with the boundedness hypothesis on solutions ought to be easy. In unbounded negative energy solutions two of the masses must form a bound pair with the third mass far away for long periods of time. During these long times the bound pair moves according to a differential equation which is a slight, but time-dependent perturbation of the Kepler equation, and so the pair should spin about each other frequently crossing the line joining their center of mass to the distant mass, thus making syzygies. However, the current author has been unable to turn this idea into a proof. The difficulties include the existence of oscillatory unbounded solutions, and the difficulty of establishing syzygies for systems looking like highly eccentric nearly Keplerian orbits subject to small time-dependent perturbations concentrated along the semi-major axis of the orbits. Instead, the methods of [**7**] have been used. The bulk of this paper is devoted to proving the existence of infinitely many syzygies for unbounded solutions with zero angular momentum. It is to be expected that a more skilled analyst could get a more direct proof based on the Kepler idea, and valid for unbound negative energy solutions with non-zero angular momentum.

1.4. *Motivation.* The current author has been trying for some time to establish a symbolical dynamical description for the zero angular momentum, three-body problem. The symbols are to be the syzygies, marked as 1, 2 and 3, depending on which mass crosses between the other two, see [**8, 9**]. A complete symbolical dynamical description has been successfully established if the potential is changed from the Newtonian $1/r$ potential to the $1/r^2$ potential and if the three masses are taken to be equal. Theorem 1.1 is a first step toward the more interesting Newtonian case. The theorem asserts that, with one exception, every solution has syzygies, and hence a symbol sequence.

2. *Proof of Theorem 1.1*

We continue to use the methods of [**7**] where we introduced the 'height' variable z on the three-body configuration space minus triple collisions. The crucial properties of this z are

$$-1 \leq z \leq 1, \tag{2.1}$$

$$|z| = 1 \iff \text{equilateral}, \tag{2.2}$$

$$z = 0 \iff \text{syzygy}, \tag{2.3}$$

and that, along any solution,

$$d(f\dot{z})/dt = -qz, \quad f > 0, \; g \geq 0, \tag{2.4}$$

where f is a smooth function on the shape space, q is a smooth function on the tangent space to the shape space,

$$q = 0 \iff \text{tangent to the Lagrange homothety} \tag{2.5}$$

and

$$f \to \infty \iff \text{unbounded}. \tag{2.6}$$

We recall that a solution is said to be bounded if all the distances r_{ij} between the pairs i, j of masses are bounded functions of time. Thus, the solution is unbounded if $\limsup r_{ij}(t) = +\infty$ for some mass pair i, j.

2.1. *The bounded case.* Let x be a solution as per the theorem and suppose it to be bounded. Thus, x is a bounded zero angular momentum, negative energy solution to the three-body problem besides the Lagrange solution. We may suppose that it is not a collinear solution since every instant of a collinear solution is a syzygy. Reflecting a solution about a line affects the transformation $z \to -z$, and a time reflection $t \to -t$ affects the transformation $\dot{z} \to -\dot{z}$. Using these symmetries and time translation, we may assume at some initial time $t = -\epsilon$ that we have $z > 0$ and $\dot{z} \leq 0$. Because the solution is not the Lagrange homothety solution, z cannot be identically 1 and q must be positive along the solution (see (2.5)), it follows from (2.4) that $(d/dt)(f\dot{z}) < 0$. In particular, $\dot{z} = 0$ identically is impossible. Upon translating time forward slightly from $-\epsilon$ to 0 we will have $\dot{z} < 0$. Now we have $z(0) > 0$ and $\dot{z}(0) < 0$. We must prove that at some finite time $t = b$ later we have $z(b) = 0$.

According to (2.4), $f\dot{z}$ is strictly decreasing as long as $z > 0$. Since f is positive, the derivative $\dot{z}$ must remain negative over any interval $[0, b)$ of time during which $z(t) > 0$. Thus, $z(t)$ is monotonically decreasing over every interval of time $[0, b)$ for which the solution exists and for which $z(t) > 0$. The solution cannot fail to exist in such an interval, because the only way it can terminate itself is by ending in a triple collision. However, all non-collinear solutions which end in a triple collision asymptote to the Lagrange solution [**6**]† implying $z \to 1$ or $z \to -1$, which we have excluded. Hence, either $b < \infty$ and $z(b) = 0$, in which case we have our syzygy, or $b = \infty$ and the solution stays in the upper hemisphere $z > 0$ for all positive time. We invoke the hypothesis that the solution is bounded to exclude the second possibility.

So, suppose that $z > 0$ on $[0, \infty)$, that $\dot{z}(0) < 0$ and that the motion is bounded. According to the bound (2.6), the function $1/f$ is bounded away from zero along our solution, so that $1/f > k$ on $[0, \infty)$ for some positive constant k. Now $\dot{z}(0)$ is negative by assumption, and $f(0)$ is positive so that $f(0)\dot{z}(0) = -a < 0$ is negative. According to the differential equation (2.4), $f\dot{z}$ is monotonically decreasing so that $f(t)\dot{z}(t) < -a$. Then $\dot{z} = (f\dot{z})/f < -ka$. But then

$$z(t) = z(0) + \int_0^t \dot{z}\, dt < z(0) - kat,$$

which violates the positivity of z as soon as $t > z(0)/ka$. This contradiction shows that in fact z has a zero before time $t = z(0)/ka$.

2.2. *The unbounded case.* There are two types of unbounded solutions, escape and oscillatory. A solution is an escape solution if $\lim r_{ij}(t) = +\infty$ for some pair ij. It is an oscillatory solution if for some pair $\limsup r_{ij} = \infty$ while for every pair $\liminf r_{ij} < \infty$. The existence of oscillatory-type unbounded solutions was established by Sitnikov [**10**]. Our proof deals simultaneously with both types.

The function $\max_{ij} r_{ij}$ is a measure of the size of the configuration. Another equivalent measure is R where $R^2 = I$ is the moment of inertia: $I = \sum m_i m_j r_{ij}^2 / \sum m_i$. The m_i are the values of the masses. Then

$$c \max_{ij} r_{ij} < R < C \max_{ij} r_{ij},$$

where here and throughout c, C denote positive constants depending only on the masses and energy. The precise values of these constants will not be important. It follows that a motion is an escape motion if $\lim_{t\to\infty} R(t) = +\infty$ and it is oscillatory if $\limsup_t R(t) = +\infty$ while $\liminf_t R(t) < \infty$. The function f of the basic equation (2.4) is related to R^2 by

$$f = R^2 \lambda^2 \tag{2.7}$$

where $c \leq \lambda \leq C$. Relation (2.6) follows from this expression for f.

† See in Table 1 in [**6**] the entry $\dim(St(R))$ with $R = C^*$. The linearization at C^* for collinear motion also has $\dim(St(R)) = 1$, showing that the stable manifold ingoing to a collinear triple collision C^* lies within the collinear submanifold.

Let x be an unbounded solution. Being unbounded, for any $R_0 > 0$ there is a time t such that $R(t) \geq R_0$. Rewrite the z equation (2.4) by introducing a new time variable σ:

$$f\frac{dz}{dt} = \frac{dz}{d\sigma} \quad \text{so that } \frac{dt}{d\sigma} = f.$$

The differential equation $(1/f)(d^2z/d\sigma^2) = -qz$ for z then becomes a harmonic oscillator:

$$\frac{d^2z}{d\sigma^2} = -\omega^2 z, \quad \omega^2 = fq \tag{2.8}$$

of variable frequency ω. If ω were to be a constant ω_0 then this would be the equation of a linear oscillator and the zeroes of any solution would be spaced equally at (σ-) intervals of length π/ω_0. Returning to our case, from standard Sturm–Liousville theory it follows that if $\omega^2 > \omega_0^2$ then within each of these intervals of length π/ω_0 the function $z(\sigma)$ has a zero. Let ℓ be the length of an interval of σ during which $R \geq R_0$ and suppose that $\omega \geq \omega_0$ during this interval. In the σ variable, escape to infinity takes a finite time so the lengths ℓ will be finite. If we can show that for R_0 sufficiently large $\ell > \pi/\omega_0$ then we will know there is an oscillation during this interval. Below, we will establish the asymptotics:

$$\ell\omega_0 \geq CR_0, \quad R_0 \to \infty. \tag{2.9}$$

It follows that there are many (at least CR_0/π) syzygies during the interval ℓ.

The following two estimates yield (2.9):

$$\omega \geq CR_0^2, \tag{2.10}$$

$$\ell \geq C/R_0. \tag{2.11}$$

2.3. *Proving estimate (2.10), the ω bound.* Let

$$r = \min_{i \neq j} r_{ij}$$

be the minimum distance. Fix the total energy H to be negative and write $h = -H > 0$. Then, as is well known, there exists a constant c depending only on the masses such that the minimum intermass distance r satisfies

$$r \leq c/(|H|). \tag{2.12}$$

See equation (A.3) of Appendix A for a proof.

The total energy is given as

$$H = (K/2) - U$$

where $K \geq 0$ is the potential energy and

$$U = \sum m_i m_j / r_{ij}$$

is the negative of the potential energy. Because our solution has negative energy and R is large for long time intervals, we know that along any one such 'long' interval one of the distances, say r_{12}, is much smaller than the other two and these other two are of order R:

$$r_{12} = r \leq C, \quad r_{13}, r_{23} \geq CR. \tag{2.13}$$

(See Appendix A, equations (A.6) and (A.8) for proofs.) Introduce the spherical coordinates (R, θ, ϕ) used in [**7**] and the squared distance variables

$$s_k = r_{ij}^2$$

for i, j, k any permutation of 123. These systems of coordinates are related by $s_k = R^2\lambda(1 - \gamma_k(\theta)\cos(\phi))$ where $\gamma_k = \cos(\theta - \theta_k)$, and $\theta = \theta_k$, $\phi = 0$ describe the location of the binary collision ray $r_{ij} = 0$; see [**7,** equation (4.3.14)]. (The angles $\theta_k - \theta_j$ between the collision rays depend on the masses.)

The function q satisfies

$$q = \text{positive} + \frac{-4\cos(\phi)}{\sin(\phi)}\frac{\partial U}{\partial \phi}.$$

This implies that the bound (2.10) will follow from the bound

$$\frac{-4\cos(\phi)}{\sin(\phi)}\frac{\partial U}{\partial \phi} \geq CR^2 \tag{2.14}$$

valid for all R large enough, together with the defining relations (2.8) and (2.7).

We proceed to establish the bound (2.14). We have

$$\frac{\partial U}{\partial \phi} = -\sum \frac{m_i m_j}{r_{ij}^3}\frac{\partial s_k}{\partial \phi}$$

and

$$\begin{aligned}\frac{\partial s_k}{\partial \phi} &= -R^2\lambda\gamma_k\sin(\phi) + R^2\frac{\partial\lambda}{\partial\phi}(1 - \gamma_k(\theta)\cos(\phi)) \\ &= \sin(\phi)R^2\lambda\gamma_k + \Lambda s_k\end{aligned}$$

where $\Lambda = d\log\lambda/d\phi$. The function λ and hence $\log\lambda$ are even functions on the shape sphere, where 'even' and 'odd' refer to behavior under the reflection$(\phi, \theta) \mapsto (-\phi, \theta)$ about the equator. It follows that Λ is an odd function and so vanishes on the equator. Thus

$$\Lambda = \sin(\phi)W(\phi, \theta),$$

where W is a smooth function on the sphere. In particular, W is uniformly bounded. Now we have

$$\frac{-1}{\sin(\phi)}\frac{1}{r_{ij}^3}\frac{\partial s_k}{\partial \phi} = \frac{1}{r_{ij}^3}[R^2\lambda\gamma_k(\theta) - Ws_k].$$

Since $\gamma_3 = 1$ at a collision and since the point in the shape sphere representing our triangle is arbitrarily close to this same collision point for R large (because $r/R \ll 1$), we have that γ_3 can be as close to 1 as we wish along our solution interval, by taking R large along the interval. Using this fact and the bound (2.12), we have

$$\begin{aligned}\frac{-1}{\sin(\phi)}\frac{1}{r_{12}^3}\frac{\partial s_3}{\partial \phi} &= \frac{1}{r_{12}^3}[R^2C - C] \\ &\geq CR^2,\end{aligned}$$

and for the other two distances

$$\left|\frac{-1}{\sin(\phi)}\frac{1}{r_{13}^3}\frac{\partial s_2}{\partial \phi}\right|, \left|\frac{-1}{\sin(\phi)}\frac{1}{r_{23}^3}\frac{\partial s_1}{\partial \phi}\right| \leq C/R.$$

Thus,

$$-4\frac{\cos(\phi)}{\sin(\phi)}\frac{\partial U}{\partial \phi} \geq C R^2$$

as claimed.

2.4. *Proving the estimate on ℓ, the bound (2.11).* We will be using the length $\rho = \|\xi\|$ of the long Jacobi vector ξ as a measure of escape. This vector connects the 12 center of mass to the distant mass three. We have

$$R^2 = a\rho^2 + br_{12}^2 \tag{2.15}$$

where a, b depend only on the masses (see (A.4) of Appendix A). It follows from equation (2.15) and the bound $r \leq c/|H|$ on $r = r_{12}$ that R and ρ are related by

$$c_a\rho \leq R \leq C_a\rho, \tag{2.16}$$

$$\frac{1}{C_a}R \leq \rho \leq \frac{1}{c_a}R, \tag{2.17}$$

where the constants c_a, C_a depend only on the masses. (These constants can be taken arbitrarily close to the constant $1/\sqrt{a}$ by taking R_0 sufficiently large and $R > R_0$.)

The desired length bound follows from the following assertion.

PROPOSITION 2.1. *Let $\rho(t)$ be the length at time t of the long Jacobi vector for a future-unbounded negative energy solution. Then there exists a constant c_3 such that for all ρ_0 sufficiently large there exist a $\rho_* \geq \rho_0$ and two times $t_1 < t_*$ such that $\rho(t_1) = \rho_*$ while $\rho(t_*) = 2\rho_*$ and ρ is monotonically increasing over the interval $t_1 < t < t_2$ with the derivative bound*

$$|\dot{\rho}(t)| \leq c_3. \tag{2.18}$$

If the solution is oscillatory, then we can take t_* such that $\dot{\rho}(t_*) = 0$ and, continuing further, find $t_3 > t_*$ such that $\rho(t_3) = \rho_*$, and ρ decreases monotonically over $[t_*, t_3]$ with the bound (2.18) in place.

In the oscillatory case, the constant c_3 can be taken to be as small as we wish. In the escape case the limit $\nu_\infty := \lim_{t\to\infty} \dot{\rho}(t)$ exists and we can take for c_3 any constant greater than ν_∞.

We show how the bounds of the proposition imply the desired bound (2.11) on ℓ. We measure the length ℓ of the domain of the arc of solution guaranteed by the proposition,

$$\begin{aligned}
\ell = \int d\sigma = \int \frac{d\sigma}{dt}dt = \int \frac{dt}{R^2\lambda} \\
\geq K\int_{t_1}^{t_*}\frac{dt}{\rho^2} = K\int_{t_1}^{t_*}\frac{dt}{d\rho}\frac{d\rho}{\rho^2} \\
\geq \frac{K}{c_3}\int_{\rho_*}^{2\rho_*}\frac{d\rho}{\rho^2} = \frac{K}{2c_3}\frac{1}{\rho_*} \\
\geq \frac{C}{R_*},
\end{aligned}$$

which is the desired bound.

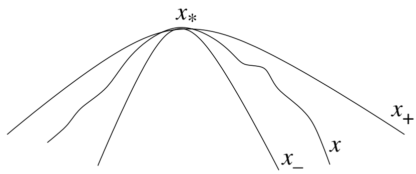

FIGURE 1. Functions in the comparison lemma.

Proof of Proposition 2.1. The proof divides into two cases, escape and oscillatory. Both cases rely on the inequality

$$-c_-/\rho^2 < \ddot{\rho} < -c_+/\rho^2, \tag{2.19}$$

valid for $\rho > \rho_0$ with ρ_0 large enough. As usual, the constants $c_- > c_+ > 0$ depend only on the masses. By taking ρ_0 arbitrarily large, we can make c_-, c_+ arbitrarily close to each other and to the total mass. See Appendix A, Lemma A.1 for the proof of (2.19).

Case 1: Escape. Say that $\rho(t) \to \infty$ with t. According to (2.19) its speed $\dot{\rho}$ decreases with increasing ρ. For t sufficiently large, we have $\dot{\rho}(t) > 0$, for otherwise we would have arbitrarily large times at which t would turn back around and ρ would decrease, contradicting escape. (See the Comparison Lemma immediately below, and Appendix A for more details.) It follows that $\dot{\rho}(t)$ is monotonically decreasing with increasing t and so tends to a limit $\nu_\infty \geq 0$. Given any $\epsilon > 0$, choose t_* large enough so that $0 < \dot{\rho} \leq \nu_\infty + \epsilon$ while $\rho(t_*) := \rho_* > \rho_0$. Then for all $t > t_*$ we have $\rho(t) \geq \rho_*$ while $0 < \dot{\rho}(t) \leq \nu_\infty + \epsilon$. Thus ρ travels between ρ_* and ∞ all the while satisfying $|\dot{\rho}| \leq c_3 = \nu_\infty + \epsilon$.

Case 2: Oscillatory. We use inequality (2.19) in conjunction with the following lemma.

LEMMA 2.2. (Comparison Lemma) *Consider three scalar differential equations* $\ddot{x}_- = F_-(x_-)$, $\ddot{x} = F(x, t)$, $\ddot{x}_+ = F_+(x_+)$ *with* F, F_+, F_- *being C functions satisfying* $F_-(x) < F(x, t) < F_+(x) < 0$ *for* $x > x_c$, *where* x_c *is a fixed constant. Suppose that* $F_-(x)$ *and* $F_+(x)$ *are monotone increasing for* $x > x_c$. *Let* $x_-(t)$, $x(t)$, $x_+(t)$ *be the solutions to their respective differential equations which share initial conditions at* $t = 0$, $x_-(0) = x_1(0) = x_+(0) := x_* > x_c$, $\dot{x}_-(0) = \dot{x}(0) = \dot{x}_+(0)$. *Then, for all times t such that* $x_-(t) \geq x_c$, *we have:*

(1) $x_-(t) \leq x(t) \leq x_+(t) \leq x_*$ *with equality only at* $t = 0$; *and*

(2) $dx_-(t)/dt < dx(t)/dt < dx_+(t)/dt$ *for* $t > 0$ *and*
$dx_-(t)/dt > dx(t)/dt > dx_+(t)/dt$ *for* $t < 0$.

See Figure 1.

We prove the lemma below. Assuming the lemma, we continue with the proof of the proposition in the oscillatory case. Let a large ρ_0 be chosen so that the estimate of (2.19) is in force for $\rho > \rho_0$, with the constants c_+, c_- sufficiently close to each other. How close is detailed in the next paragraph. Since the solution is oscillatory, given any $\rho_* > 0$ we can find times t_* arbitrarily large such that

$$\rho(t_*) := 2\rho_*$$

and

$$\dot{\rho}(t_*) = 0.$$

Since ρ_* is arbitrarily large, we may suppose that $\frac{1}{2}\rho_* \geq \rho_0$. The Comparison Lemma sandwiches ρ between the solutions ρ_+, ρ_- to the 'bounding' differential equations: $\ddot{\rho}_\pm = -c_\pm/\rho_\pm^2$ of (2.19), which share initial conditions with ρ at $t = t_*$. See Figure 1. Thus,

$$\rho_-(t) < \rho(t) < \rho_+(t), \quad t \in I,$$

where I is an interval containing t_* such that for $t \in I$ the bound $\rho(t) > \rho_0$ needed to obtain (2.19) is in force.

We can describe the comparison solutions $\rho_\pm$ in sufficient detail by using the scaling symmetry of Kepler's equation. Let $\phi(t)$ be the solution to the model Kepler equation $\ddot{\phi} = -1/\phi^2$ with initial conditions $\phi(0) = 1$, $\dot{\phi}(0) = 0$. Then

$$\rho_+(t) = \lambda\phi(\lambda^{-3/2}\sqrt{c_+}(t - t_*))$$

and

$$\rho_-(t) = \lambda\phi(\lambda^{-3/2}\sqrt{c_-}(t - t_*))$$

where we take

$$\lambda = 2\rho_*$$

to guarantee agreement of initial conditions at $t = t_*$. By taking ρ_0 sufficiently large we can make c_- arbitrarily close to c_+. Consequently, for ρ_0 large enough we will have that $1/4 \leq \phi(\sqrt{c_-}\tau_f)$ where $\tau_f > 0$ is a time such that $\phi(\sqrt{c_+}\tau_f) = 1/2$. Now the times τ and t for the scaled solutions are related by $\tau = \lambda^{-3/2}(t - t_*)$. It follows that at the time t_2 corresponding to τ_f we have

$$\rho_+(t_f) = \rho_*$$

and $\rho_0 < \rho_*/2 < \rho_-(t_f) < \rho(t_f) \leq \rho_*$. Over the time interval $[0, \tau_f]$ the uniform derivative bound $-k < \sqrt{c_-}(d\phi/d\tau)(\sqrt{c_-}\tau) < 0$ holds. Under the scaling and translation symmetry used to make $\rho_\pm$ we find that the velocities transform by

$$v(t) = \frac{1}{\sqrt{\lambda}}\frac{d\phi}{d\tau}(\lambda^{-3/2}(t - t_*)).$$

Consequently, $k/\sqrt{\lambda} < \dot{\rho}_- < 0$ during the time interval $[t_*, t_f]$. By the Comparison Lemma, then

$$-k/\sqrt{2\rho_*} < \dot{\rho} < 0$$

over this same time interval. Now $\rho(t_f)$ may be less than ρ_* but ρ is monotonically decreasing. So take for t_3 the unique time in the interval $[t_*, t_f]$ such that $\rho(t_3) = \rho_*$. This completes the argument in the oscillating case for the decreasing interval $[t_*, t_3]$ of ρ. The argument for the increasing arc $[t_2, t_*]$ of ρ is the time reversal (about t_*) of this argument.

□

Proof of Lemma 2.2. (1) follows from (2) by integration. We will just prove (2) in the $-$ case, i.e. the inequality $dx_-/dt < dx/dt$ for $t > 0$. The argument in the other cases is identical. Looking at the Taylor expansions of x_-, x at $t = 0$, we see that the inequality holds in a small right-hand neighborhood of 0, say $(0, \delta)$. Now proceed by contradiction. If the inequality fails before x_- reaches x_c, then there is a t with $dx_-(t)/dt \geq dx(t)/dt$. Let t_* be the first such $t > 0$ such that $dx_-(t)/dt = dx/dt$. We have $t_* > \delta$. By integration, $dx_-(t_*)/dt = \int_0^{t_*} F_-(x_1(t))\, dt$ and $dx(t_*)/dt = \int_0^{t_*} F(x(t), t)\, dt$. These two integrals are equal. But $F(x(t), t) > F_-(x(t))$. Moreover, in the interval $(0, t_*)$ we have $dx_-/dt < dx/dt$ and so, by integration, $x_-(t) < x(t)$. Then $F_-(x(t)) > F_-(x_-(t))$ by the monotonicity of F_-. So $\int_0^{t_*} F(x(t), t)\, dt > \int_0^{t_*} F_-(x_-(t))\, dt$, contradicting the equality of the two integrals. □

Remark. Differential inequalities involving ρ are much better behaved at large R (and hence ρ) than those involving R. The R differential equation is the Lagrange–Jacobi identity $2d(R\dot{R})/dt = 4H + 2U$ and yields a huge second derivative for R when $r = r_{12}$ is sufficiently small. Thus R can oscillate wildly, despite the fact that the bound (2.15) is in force.

3. *Discussion—open questions*

Could Theorem 1.1 hold for arbitrary energy H and angular momentum J? No. It does not hold for $H > 0$ and $J = 0$. The direct method of the calculus of variations yields action-minimizing hyperbolic escape orbits, which leave triple collisions and tend to any desired non-collinear point of the shape sphere in infinite time. The reflection argument (see, e.g., [**5**]) shows that these minimizers never become collinear. The theorem might hold for $H = J = 0$ but we suspect not. In this case there is a manifold of parabolic escape orbits whose shapes tend to Lagrange. It is believed that some of these have no syzygies, but this is just a guess. For $H < 0$ and general $J \neq 0$ the theorem is false, at least for mass ratios in which one mass dominates. In this case, the near-circular Lagrange solutions are KAM (Kolmogorov–Arnold–Moser) stable (see [**4**]), and so are surrounded by a nearby cloud of KAM tori on which the solutions stay near the Lagrange solution, and hence away from $z = 0$ for all time. It is possible that for some values of $H < 0$, $J \neq 0$ and some values of the mass ratios that the theorem continues to hold. If the Dziobek constants $J^2 H$ and mass ratios are such that the Lagrange solution is unstable (which is the case for nearly equal masses and $J^2 H$ being a value near that which supports the circular Lagrange solution), then there is some chance for the theorem to hold.

According to the theorem, all solutions bi-asymptotic to a triple collision except for the Lagrange solution have syzygies. This number is necessarily finite. What numbers are possible? Is any finite number of syzygies achieved? Is any finite syzygy sequence realized? Write $m(x)$ for the time interval on which the solution x is defined. (Thus, $m(x) = +\infty$ for all solutions except those bi-asymptotic to a triple collision.) Is it true that $m(x)$ is minimized (among all solutions x with $J = 0$ and $H < 0$ fixed) by the Lagrange solution?

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A. *Appendix. Bounds near ∞ for negative energy*

We suppose that the total energy $H = K/2 - U$ is negative and write $H = -|H|$. Then

$$U \geq |H|. \tag{A.1}$$

Write $r = \min\{r_{ij} : i \neq j\}$ for the minimum of the intermass distances. Then there is a constant c depending only on the masses such that

$$c/r \geq U. \tag{A.2}$$

(For instance, if the masses are all equal to m then $c = 3m^2$.) It follows that

$$c/r \geq |H|$$

or

$$c/|H| \geq r. \tag{A.3}$$

Let us suppose that 12 realize the minimum distance:

$$r = r_{12}.$$

Associated to the decomposition 12; 3 we have Jacobi vectors and their lengths:

$$\zeta = x_1 - x_2, \quad |\zeta| = r,$$
$$\xi = x_3 - x_{cm}^{12}, \quad |\xi| = \rho.$$

Here

$$x_{cm}^{12} := (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$$
$$:= \mu_1 x_1 + \mu_2 x_2$$

is the 12 center of mass and

$$\mu_1 = m_1/(m_1 + m_2), \quad \mu_2 = m_2/(m_1 + m_2).$$

One computes

$$R^2 = \alpha_1 r^2 + \alpha_3 \rho^2 \tag{A.4}$$

where

$$\alpha_1 = m_1 m_2/(m_1 + m_2), \quad \alpha_3 = (m_1 + m_2) m_3/(m_1 + m_2 + m_3), \tag{A.5}$$

from which it follows that

$$c_a \rho \leq R \leq C_a \rho \tag{A.6}$$

and

$$U \geq C/\rho. \tag{A.7}$$

Set $\hat{U} = RU$. Combine (A.1), (A.2), (A.7) with (A.6) to get that $C\rho/r \geq \hat{U} \geq R|H|$ or

$$\frac{C}{R|H|} \geq r/\rho,$$

which asserts that by making R or ρ large we can make the ratio r/ρ as small as we wish. We view r/ρ as a perturbation parameter. From the last inequality it follows that for every $\epsilon > 0$ there is a ρ_0 (or R_0) sufficiently large such that $\rho \geq \rho_0$ ($R \geq R_0$) implies that $r/\rho < \epsilon$. In what follows, let small ϵ be given, and suppose we have chosen the corresponding ρ_0 (or R_0) to be taken so as to guarantee $r/\rho < \epsilon$. Moreover, let $c, C, \ldots$ denote constants depending only on this ρ_0, the masses, the total energy and the total angular momentum.

We have $r = r_{12} = |\zeta|$. To express the other distances in terms of ξ, ζ, use $x_3 - x_1 = \xi - \mu_2\zeta$ and $x_3 - x_2 = \xi + \mu_1\zeta$ to obtain

$$r_{13} = \|\xi - \mu_2\zeta\|, \qquad r_{23} = \|\xi + \mu_1\zeta\|, \tag{A.8}$$

where μ_i are the reduced masses described above. Note that $r_{13}, r_{23} = \rho + O(\epsilon)$.

Then

$$H = H_{12} + H_3 + g$$

where

$$H_{12} = \tfrac{1}{2}\alpha_1\|\dot{\zeta}\|^2 - \beta_1/r,$$
$$H_3 = \tfrac{1}{2}\alpha_3\|\dot{\xi}\|^2 - \beta_3/\rho,$$

where α_1, α_2 are given in equation (A.5),

$$\beta_1 = m_1 m_2, \qquad \beta_3 = (m_1 + m_2)m_3$$

and the 'error term' g is given by

$$g = \frac{(m_1 + m_2)m_3}{\|\xi\|} - \frac{m_1 m_3}{\|\xi - \mu_2\zeta\|} - \frac{m_2 m_3}{\|\xi + \mu_1\zeta\|}.$$

Recall that the Legendre polynomials $P_j(x)$ can be defined by

$$\begin{aligned}\frac{1}{\|\xi - q\|} &= \frac{1}{\|\xi\|}\left\{1 + \frac{\|q\|}{\|\xi\|}\cos(\psi) + \left(\frac{\|q\|}{\|\xi\|}\right)^2 P_2(\cos\psi)\right.\\ &\quad \left. + \cdots + \left(\frac{\|q\|}{\|\xi\|}\right)^j P_j(\cos\psi) + \cdots\right\}\\ &= \frac{1}{\|\xi\|} + \frac{P_1(\xi\cdot q)}{\|\xi\|^3} + \frac{P_2(\xi\cdot q)}{\|\xi\|^5} + \cdots + \frac{P_j(\xi\cdot q)}{\|\xi\|^{2j+1}} + \cdots,\end{aligned}$$

where ψ is the angle between ξ and q. (Note that $P_1(x) = x$.) The P_j are homogeneous degree j polynomials. Expanding each of the three terms of g in terms of Legendre polynomials yields a Laurent series in $\rho = \|\xi\|$. The first two terms of this series, i.e. the $1/\rho$ term and the $1/\rho^2$ term, are zero because $(m_1 + m_2)m_3 = m_1 m_3 + m_2 m_3$ and because $m_1\mu_2 - m_2\mu_1 = 0$. We then find that

$$\begin{aligned}g &= \left(\frac{m_3}{\rho}\right)\left\{(m_1\mu_2^2 + m_2\mu_1^2)\left(\frac{r}{\rho}\right)^2 P_2(\cos\psi) + O\left(\frac{r}{\rho}\right)^3\right\}\\ &= \frac{(m_1\mu_2^2 + m_2\mu_1^2)P_2(\xi_1\cdot\xi_2)}{\rho^5} + O(r^3/\rho^4).\end{aligned}$$

We will need estimates for g and its derivatives. These are

$$|g| \leq C\epsilon^2/\rho \tag{A.9}$$

or

$$|g| \leq Cr^2/\rho^3 \tag{A.10}$$

and

$$g_{\xi_1} = O(\rho^{-3})$$

and

$$g_{\xi_2} = O(\rho^{-4})$$

and so on.

If we set $g = 0$ then H becomes the Hamiltonian for two uncoupled Kepler systems. The next lemma describes some details of the asymptotics of this decoupling as $\rho \to \infty$. Introduce

$$J_{12} = \mu_1 \zeta \wedge \dot{\zeta},$$

the angular momentum of the 12 system, and the radial and transverse separation velocities ν, $V^\perp$ given by

$$\dot{\xi} = \nu\hat{\xi} + V^\perp, \quad \dot{\rho} = \nu,$$

where $\hat{\xi} = \xi/\rho$ is the unit vector in the ξ direction and where $V^\perp$ is orthogonal to ξ.

LEMMA A.1. *Consider any solution to the three-body problem along which $\rho(t) \geq \rho_0$ with ρ_0 as above. There exists a positive constant c, depending only on the total energy H and total angular momentum J, the masses m_i and ρ_0, such that:*

(a) $|J_{12}| \leq c$;

(b) $\|V^\perp\| \leq c/\rho$;

(c) $|\ddot{\rho} + M/\rho^2| \leq \epsilon/\rho^3$ *where $M = m_1 + m_2 + m_3$ is the total mass.*

Similar bounds hold for the time derivatives of H_{12}, H_3, J_3, $\hat{\xi}$, and the 12 and 3 Laplace (or Runge–Lenz) vectors.

Proof. Estimate (a): We show that $\|J_{12}\| \leq \beta_1^2/|H| + O(1/\rho_0)$. We have $J_{12} = \alpha_1 \zeta \wedge \dot{\zeta}$, $\|\zeta \wedge \dot{\zeta}\|^2 \leq \|\zeta\|^2\|\dot{\zeta}\|^2$ and $|\zeta|^2 = r^2$. It follows that

$$|J_{12}| \leq \alpha_1 r^2 \|\dot{\zeta}\|^2.$$

Set $H' = H_3 + g$. Note that $-H' \leq \beta_3/\rho + \epsilon/\rho \leq c/\rho_0$, and that $H_{12} = H - H' \leq -|H| + c/\rho_0$. Use the formula for H_{12} to rewrite this last inequality as

$$\alpha_1 \|\dot{\zeta}\|^2 \leq -2|H| + 2c/\rho_0 + 2\beta_1/r.$$

Multiply through by r^2 to get

$$|J_{12}| \leq \alpha_1^2 r^2 \|\dot{\zeta}\|^2 \leq [-2|H| + 2c/\rho_0]r^2 + 2\beta_1 r.$$

The right-hand side is a quadratic function of r with negative quadratic terms. The maximum value of this quadratic function is $(\frac{1}{2})(2\beta_1)^2/(2|H| - 2c/\rho_0) = \beta_1^2/|H| + O(1/\rho_0)$. Thus $|J_{12}| \leq \beta_1^2/|H| + O(1/\rho_0)$.

Estimate (b): We have $\alpha_3 \xi \wedge \dot{\xi} = \alpha_3 \xi \wedge V^\perp$, $|\xi \wedge V^\perp| = \rho|V^\perp|$ and $\alpha_3 \xi \wedge \dot{\xi} = J - J_{12}$. Thus,

$$\begin{aligned} \alpha_3 \rho |V^\perp| &= |J - J_{12}| \\ &\leq |J| + |J_{12}|. \end{aligned}$$

Now use estimate (a).

Estimate (c): We have $\dot{\rho} := \langle \hat{\xi}, \dot{\xi} \rangle$, so that

$$\ddot{\rho} = \left\langle \frac{d}{dt}\hat{\xi}, \dot{\xi} \right\rangle + \langle \hat{\xi}, \ddot{\xi} \rangle. \tag{A.11}$$

We compute that

$$\left\langle \frac{d}{dt}\hat{\xi}, \dot{\xi} \right\rangle = -\frac{\dot{\rho}^2}{\rho} + \frac{\|\dot{\xi}^2\|}{\rho} = \|V^{\perp}\|^2/\rho$$

so that by estimate (b)

$$\left| \left\langle \frac{d}{dt}\hat{\xi}, \dot{\xi} \right\rangle \right| \leq c/\rho^3. \tag{A.12}$$

Now use Newton's equations for ξ,

$$\alpha_3 \ddot{\xi} = U_\xi = -\beta_3 \xi/\rho^3 + g_\xi,$$

which yields

$$\ddot{\xi} = -M\xi/\rho^3 + \frac{1}{\alpha_3} g_\xi$$

because $\beta_3/\alpha_3 = M$. Thus,

$$\langle \hat{\xi}, \ddot{\xi} \rangle = -M/\rho^2 + \frac{1}{\alpha_3} \langle \hat{\xi}, g_\xi \rangle. \tag{A.13}$$

Using the estimate (A.10) on the gradient g_ξ of g, with equations (A.11), (A.12) and (A.13), we get the desired result $|\ddot{\rho} + M/\rho^2| \leq c/\rho^3$. □

As a general reference for some of the inequalities appearing here, and many others, see Marchal [**4,** especially equations (885)–(894)].

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