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THE HAMILTONIAN STRUCTURE FOR DYNAMIC FREE BOUNDARY PROBLEMS

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Hamiltonian structures for 2- or 3-dimensional incompressible flows with a free boundary are determined which generalize a previous structure of Zakharov for irrotational flow. Our Poisson bracket is determined using the method of Arnold, namely reduction from canonical variables in the Lagrangian (material) description. Using this bracket, the Hamiltonian form for the equations of a liquid drop with a free boundary having surface tension is demonstrated. The structure of the bracket in terms of a reduced cotangent bundle of a principal bundle is explained. In the case of two-dimensional flows, the vorticity bracket is determined and the generalized enstrophy is shown to be a Casimir function. This investigation also clears up some confusion in the literature concerning the vorticity bracket, even for fixed boundary flows.

1. Introduction

This paper determines the Poisson bracket structure for an incompressible fluid with a free boundary and shows that the equations for an ideal fluid having a free boundary with surface tension are Hamiltonian relative to this structure. The Poisson bracket structure we derive generalizes that found in the irrotational case by Zakharov [1]; see also Miles [2], Benjamin and Olver [3] and references therein. Our aim is not merely to exhibit the bracket but rather to understand its derivation and its geometric structure.

The method we use to obtain the Poisson bracket structure is to pass from canonical brackets in the Lagrangian (material) representation to non-canonical brackets in the Eulerian (spatial) representation by eliminating the gauge symmetry of particle relabelling. This method, going back to Arnold [4], is at the basis of the general theory of reduction (Marsden and Weinstein [5]) and was used by Marsden and Weinstein [6] to derive the bracket structure for the Maxwell–Vlasov equations and the equations for incompressible flow with fixed boundaries.

We shall give two representations for the Poisson bracket. The first, and most elementary, form is given in section 2. This has the structure of a Lie–Poisson bracket (see Marsden and Weinstein [6] or Marsden et al. [7] for background and references on Lie–Poisson structures) plus a canonical bracket, although the variables used in these two terms are not independent. The second representation, given in section 4, gives the bracket as a special case of the Poisson bracket on the reduction of the cotangent bundle of a principal bundle by its group due to Montgomery, Marsden and Ratiu [8] (see also Kummer [9]). These brackets have the following general structure (schematically):

$$\{F,G\} = (\text{Lie–Poisson bracket}) + (\text{Curvature term}) + (\text{Canonical bracket}).$$

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In other examples, the curvature term can represent Coriolis or magnetic forces (see for example, Kummer [9] and Krishnaprasad and Marsden [10]). In our example the Lie–Poisson bracket represents the internal fluid contribution decoupled from the canonical bracket for the boundary motion. The coupling between the fluid and the boundary is now carried by the curvature term. In either representation the canonical bracket is the term corresponding to the bracket of Zakharov.

The two representations of the bracket are sometimes called the "Weinstein" and "Sternberg" representations since they correspond to two Hamiltonian representations of a particle in a Yang–Mills field found by these authors. (See Guillemin and Sternberg [11] and references therein.) It was this work which led, via Montgomery [12], to the general principal bundle picture of Montgomery, Marsden and Ratiu [8].

In view of the detailed understanding of this case, we expect that one can similarly obtain brackets for free boundary problems for compressible flow and plasmas, either relativistic or not. This will clearly involve semidirect products in the Lie–Poisson part, as in Montgomery, Marsden and Ratiu [8]. For papers which explicate and review bracket structures for these other problems, see, for example, the articles in Marsden [13] and for the relativistic case, see Holm and Kupershmidt [14], Bao et al. [15], Holm [16], Marsden et al. [7] and references therein.

We expect that the Hamiltonian structure studied here will be useful for a variety of questions, including the following:

1) Nonlinear stability of equilibria; see Arnold [4, 17], Sedenko and Iudovich [18], Artale and Salusti [19], papers in Marsden [13], Holm, Marsden, Ratiu, and Weinstein [20] and Abarbanel, Holm, Marsden, and Ratiu [21].

2) Short time existence, uniqueness, smoothness, and convergence theorems using the method of Ebin and Marsden [22].

3) Bifurcations of rotating liquid drops (Brown and Scriven [23]).

4) A study of the modulation equations and relationships to other surface wave models (Zakharov [1], Olver [24]).

5) A study of prechaotic motion in the forced vibration of a fluid with a free surface (Benjamin and Ursell [25], Miles [26]) using the Melnikov method (Holmes and Marsden [27] and Holmes [28]).

With a view towards item 1), we show that for two dimensional ideal flow, the generalized enstrophy is a Casimir function. In this connection, we show (contrary to what is often stated) that the Poisson bracket for two dimensional flow, even with a *fixed* boundary, is *not*

$$\{F,G\} = \int_D \omega \left\{ \frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right\}_{xy} \mathrm{d}x\,\mathrm{d}y, \tag{1.1}$$

where

$$\{f,g\}_{xy} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y},$$

D is the flow region and $\delta F/\delta\omega$ is interpreted in the usual way, but rather needs to be corrected by the addition of a boundary term. (This does not contradict a similar looking and correct formula given in Marsden and Weinstein [6] because they interpret $\delta F/\delta\omega$ in a more sophisticated way.) This boundary correction to (1.1) is necessary so that the generalized enstrophy

$$C(\omega) = \int_D \Phi(\omega)\,\mathrm{d}x\,\mathrm{d}y \tag{1.2}$$

is a Casimir function.

The plan of the paper is as follows. In section 2 we state the first version of our Poisson bracket and verify that the equations for a liquid drop with surface tension are Hamiltonian. In section 3 we derive this Poisson bracket by reduction of canonical brackets from the Lagrangian description and in section 4 we present a second representation of the bracket and explain how it is a special case of the bracket on the reduction of the cotangent bundle of a principal bundle by its structure group. Finally, in section 5, we present the corrected vorticity bracket for two dimensional flow and check that the generalized enstrophy is indeed a Casimir function.

In this paper we do not attempt to make precise all the function spaces needed for a proper analytical treatment of the infinite dimensional manifolds involved. Most of this can be filled in routinely following Ebin and Marsden [22] (see also Cantor [29] for the non-compact case). These analytical aspects properly belong with a detailed investigation of existence and uniqueness questions, and so are deferred to a later study.

2. The first Poisson bracket and the equations for a liquid drop

We shall first state the Poisson bracket for the free boundary problem and then shall show that the equations for a liquid drop with a free boundary and surface tension are Hamiltonian relative to this bracket. The derivation of the bracket is given in the next section.

The basic dynamic variables we use are the spatial velocity field $\boldsymbol{v}$ and the free surface Σ. We assume that $\boldsymbol{v}$ is divergence free and is defined on D_Σ, the region whose boundary is Σ. (Corresponding to $\boldsymbol{v}$ being divergence free, D_Σ will have constant volume.) The surface Σ is assumed to be compact and diffeomorphic to the boundary of a reference region D. We take Σ to be unparametrized. Thus, Σ is a 2-manifold in $\mathbf{R}^3$ (or a curve in $\mathbf{R}^2$ for planar flow); it is not a map of ∂D to $\mathbf{R}^3$, but rather is the image of such a map*.

According to Weyl–Hodge theory (see Ebin and Marsden [22] for a summary and references), $\boldsymbol{v}$ decomposes uniquely as

$$\boldsymbol{v} = \boldsymbol{w} + \nabla\Phi, \tag{2.1}$$

where $\boldsymbol{w}$ is divergence free and tangent to Σ. Notice that Φ is determined (up to an additive constant) by

$$\nabla^2\Phi = 0, \quad \frac{\partial\Phi}{\partial\nu} = \langle \boldsymbol{v}, \boldsymbol{\nu}\rangle, \tag{2.2}$$

*We have also worked out the bracket for the case in which Σ is parametrized, but the theory seems superior in the unparametrized case and makes more direct contact with the existing literature.

where $\boldsymbol{\nu}$ is the unit outward normal to Σ and $\langle\, ,\rangle$ is the inner product on $\mathbf{R}^3$.

Let $\mathscr{N}$ be the space of pairs $(\boldsymbol{v}, \Sigma)$. The space $\mathscr{N}$ will be the basic phase space for the first representation of the bracket; the other representation will be in terms of the set $\mathscr{N}'$ of triples $(\boldsymbol{w}, \phi, \Sigma)$, where ϕ is the restriction of Φ to Σ and is understood to be taken modulo additive constants.

The bracket will be defined for functions F, G: $\mathscr{N} \to \mathbf{R}$ which possess functional derivatives, defined as follows:

a) $\delta F/\delta \boldsymbol{v}$ is a divergence free vector field on D_Σ such that for all variations $\delta\boldsymbol{v}$

$$\mathrm{D}_{v}F(\boldsymbol{v}, \Sigma)\cdot\delta\boldsymbol{v} = \int_{D_\Sigma}\left\langle \frac{\delta F}{\delta \boldsymbol{v}}, \delta\boldsymbol{v}\right\rangle \mathrm{d}^3x, \tag{2.3}$$

where D_vF is the derivative of F holding Σ fixed.

b) $\delta F/\delta\phi$ is the function on Σ with zero integral given by

$$\frac{\delta F}{\delta\phi} = \left\langle \frac{\delta F}{\delta \boldsymbol{v}}, \boldsymbol{\nu}\right\rangle. \tag{2.4}$$

(One easily checks that $\delta F/\delta\phi$ is just the variational derivative of F taken with respect to variations of $\boldsymbol{v}$ by potential flows.)

c) The definition of $\delta F/\delta\Sigma$ is slightly more involved. A variation $\delta\Sigma$ of Σ is identified with a function on Σ; it represents an infinitesimal variation of Σ in its normal direction. By the incompressibility assumption, $\delta\Sigma$ has zero integral on Σ, a condition dual to the additive constant ambiguity in ϕ. Smoothly extend $\boldsymbol{v}$ so it is defined in a neighborhood of Σ; thus, holding $\boldsymbol{v}$ constant while varying Σ makes sense. Then set

$$\mathrm{D}_\Sigma F(\boldsymbol{v}, \Sigma)\cdot\delta\Sigma = \int_\Sigma \frac{\delta F}{\delta\Sigma}\delta\Sigma\,\mathrm{d}A, \tag{2.5}$$

so $\delta F/\delta\Sigma$ is determined up to an additive constant. (One checks that $\delta F/\delta\Sigma$ is independent of the way $\boldsymbol{v}$ is extended, as long as F is C^1 as $\boldsymbol{v}$ varies in the C^1 topology.)

Here then is the first representation of the bracket:

Definition 2.1. Given F and G mapping $\mathscr{N}$ to $\mathbb{R}$ and possessing the functional derivatives just defined, set

$$\{F,G\} = \int_{D_\Sigma} \left\langle \omega, \frac{\delta F}{\delta v} \times \frac{\delta G}{\delta v} \right\rangle \mathrm{d}^3 x + \int_\Sigma \left(\frac{\delta F}{\delta \Sigma}\frac{\delta G}{\delta \phi} - \frac{\delta G}{\delta \Sigma}\frac{\delta F}{\delta \phi} \right) \mathrm{d}A, \tag{2.6}$$

where $\omega = \nabla \times v$ is the vorticity.

Proposition 2.2. This bracket makes $\mathscr{N}$ into a Poisson manifold; i.e. $\{\,,\}$ is real bilinear, antisymmetric, satisfies Jacobi's identity and is a derivation in F and G.

The validity of this proposition will be clear from its construction via reduction, which is given in the next section. The only nonobvious property is, of course, Jacobi's identity. Notice that the two terms in (2.6) are coupled via the definition of $\delta F/\delta\phi$. Our second representation will decouple these at the expense of introducing additional terms.

Remarks. 1) For irrotational flow $\omega = \mathbf{0}$, so (2.6) reduces to the canonical bracket in ϕ and Σ. This shows that for irrotational flow the bracket reduces to that of Zakharov [1].

2) For some functionals, such as the generalized enstrophy, the functional derivatives do not exist as we have defined them. Rather, they have contributions concentrated on Σ arising from v variations. Such terms often appear as boundary terms after an integration by parts. This situation complicates (2.8) somewhat, and will be discussed in section 5.

3) The bracket (2.6) is purely kinematical in the sense that it can be used for a variety of dynamic problems with different Hamiltonians.

To illustrate the relevance of the bracket (2.6) we consider the equations for a liquid drop consisting of an ideal (incompressible, inviscid) fluid with a free boundary and forces of surface tension on the boundary. In terms of the variables already presented, the equations are

$$\left.\begin{aligned} &\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p, \\ &\frac{\partial \Sigma}{\partial t} = \langle v, \nu \rangle, \\ &\operatorname{div} v = 0, \quad p|\Sigma = \tau\kappa, \end{aligned}\right\} \tag{2.7}$$

where κ is the mean curvature of the surface Σ and τ is the surface tension, a constant. The Hamiltonian is taken to be

$$H = \tfrac{1}{2}\int_{D_\Sigma} |v|^2 \,\mathrm{d}^3 x + \tau \int_\Sigma \mathrm{d}A. \tag{2.8}$$

Proposition 2.3. The equations (2.7) are equivalent to

$$\dot{F} = \{F, H\}$$

for all functions F (possessing functional derivatives), where H is given by (2.8) and the bracket by (2.6).

Proof. The functional derivatives of H are computed to be

$$\frac{\delta H}{\delta v} = v, \quad \frac{\delta H}{\delta \phi} = \left\langle \frac{\delta H}{\delta v}, \nu \right\rangle = \langle v, \nu \rangle,$$

and

$$\frac{\delta H}{\delta \Sigma} = \frac{1}{2}|v|^2 + \tau\kappa$$

(where $\delta H/\delta\Sigma$ is taken modulo additive constants). Now

$$\dot{F} = \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta v}, \frac{\partial v}{\partial t} \right\rangle \mathrm{d}^3 x + \int_\Sigma \frac{\delta F}{\delta \Sigma}\frac{\partial \Sigma}{\partial t}\,\mathrm{d}A \tag{2.9}$$

and from (2.6),

$$\{F,H\} = \int_{D_\Sigma}\left\langle \omega, \frac{\delta F}{\delta v}\times v\right\rangle \mathrm{d}^3x + \int_\Sigma\left\langle \frac{\delta F}{\delta \Sigma} v, \nu\right\rangle \mathrm{d}A - \int_\Sigma\left[\frac{1}{2}|v|^2+\tau\kappa\right]\frac{\delta F}{\delta\phi}\,\mathrm{d}A$$

$$= \int_{D_\Sigma}\left\langle \frac{\delta F}{\delta v}, v\times\omega\right\rangle \mathrm{d}^3x + \int_\Sigma\left\langle \frac{\delta F}{\delta \Sigma} v, \nu\right\rangle \mathrm{d}A - \int_{D_\Sigma}\left\langle \nabla\left(\frac{1}{2}|v|^2\right), \frac{\delta F}{\delta v}\right\rangle \mathrm{d}^3x - \int_\Sigma\left\langle \tau\kappa\frac{\delta F}{\delta v}, \nu\right\rangle \mathrm{d}A. \tag{2.10}$$

If (2.7) holds, then (2.9) and (2.10) are clearly equal in view of the vector identity

$$-(v\cdot\nabla)v = v\times\omega - \tfrac{1}{2}\nabla|v|^2.$$

Conversely, if (2.9) and (2.10) are equal for all $\delta F/\delta v$ and we define p to be the solution to the Dirichlet problem

$$p|\Sigma = \tau\kappa, \quad \nabla^2 p = -\operatorname{div}((v\cdot\nabla)v)$$

then the boundary term $\int_\Sigma\langle\tau\kappa\delta F/\delta v, \nu\rangle\,\mathrm{d}A$ drops out of (2.10) when ∇p is subtracted from $v\times\omega - \nabla\frac{1}{2}|v|^2$. Thus (2.7) holds. ■

3. Derivation of the bracket by reduction from the Lagrangian representation

We choose as the configuration space $\mathscr{C} = \mathrm{Emb}_{\mathrm{vol}}(D, \mathbf{R}^n)$, the manifold of volume-preserving embeddings of an n-dimensional reference manifold D, an open subset of $\mathbf{R}^n$ with smooth boundary, into $\mathbf{R}^n$. The corresponding phase space is its L_2 cotangent bundle $\mathrm{T}^*\mathscr{C} = \mathrm{T}^*\mathrm{Emb}_{\mathrm{vol}}(D, \mathbf{R}^n)$ elements of which are pairs (η, μ) where $\eta: D \to \mathbf{R}^n$ is an element of $\mathscr{C}$ (configuration maps) and μ, the momentum density, is a divergence free one form over η; i.e. to each reference point $X \in D$, μ assigns a one form on $\mathbf{R}^n$ based at the spatial point $x = \eta(X)$. The pairing of μ with a tangent vector $\delta\eta \in \mathrm{T}_\eta\mathscr{C}$, a map of D to $\mathbf{TR}^n$ which sends a reference point X to a tangent vector in $\mathbf{R}^n$ based at $x = \eta(X)$, is given by

$$\langle\langle \mu, \delta\eta\rangle\rangle = \int_D \mu(X)\cdot\delta\eta(X)\,\mathrm{d}V, \tag{3.1}$$

where the natural contraction produces a function on D which is then integrated over D with respect to $\mathrm{d}V = \mathrm{d}^3X$, the Euclidean volume element. (For compressible flow, μ should be taken to be a one form over η, tensored with a density on D.) This choice of configuration and phase space for the Lagrangian description of continuum mechanics is standard; see, for example, Marsden and Hughes [30].

Before defining the Poisson bracket on $\mathrm{T}^*\mathscr{C}$, we define the partial Fréchet and functional derivatives of a function $F: \mathrm{T}^*\mathscr{C} \to \mathbf{R}$. The partial Fréchet derivative with respect to μ is simply the fiber derivative: a variation $\delta\mu$ is also a one form over η, so the partial μ derivative is defined as usual:

$$\mathrm{D}_\mu F(\eta,\mu)\cdot\delta\mu = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} F(\eta, \mu+\varepsilon\delta\mu). \tag{3.2}$$

To define the partial derivative with respect to η one must in some sense "fix" μ while allowing η to vary; this may be accomplished, analogous to what we did in section 2, in the following fashion. We identify $\mathrm{T}^*(\eta(D))$ with $\eta(D)\times\mathbf{R}^{n*}$ and let $\tilde\mu$ denote the principal part of μ; i.e. the projection of μ onto $\mathbf{R}^{n*}$. Thus, $\tilde\mu: D \to \mathbf{R}^{n*}$ is a map such that $\mu = \eta\times\tilde\mu$: $D \to \eta(D)\times\mathbf{R}^{n*} = \mathrm{T}^*(\eta(D))$. Given a variation $\delta\eta$ and a curve η_ε tangent to $\delta\eta$ at $\varepsilon = 0$, let $\mu_\varepsilon = \eta_\varepsilon\times\tilde\mu$ and let

$$\mathrm{D}_\eta F(\eta,\mu)\cdot\delta\eta = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} F(\eta_\varepsilon, \mu_\varepsilon). \tag{3.3}$$

A function $F: \mathrm{T}^*\mathscr{C} \to \mathbf{R}$ is said to have partial functional derivatives if there exist, for every element (η,μ) of $\mathrm{T}^*\mathscr{C}$, the following:

(i) $(\delta^\wedge F/\delta\eta)(\eta,\mu)$, a divergence free one form over η;

(ii) $(\delta^{\vee}F/\delta\eta)(\eta,\mu)$, a density on ∂D tensored with a one form on D over η defined at points of ∂D;

(iii) $(\delta^{\wedge}F/\delta\mu)(\eta,\mu)$, a divergence free vector field on D over η;

(iv) $(\delta^{\vee}F/\delta\mu)(\eta,\mu)$, a vector field on D over η defined at points of ∂D tensored with a density on ∂D, satisfying

$$\mathrm{D}_\eta F(\eta,\mu)\cdot\delta\eta=\int_D\frac{\delta^{\wedge}F}{\delta\eta}(\eta,\mu)\cdot\delta\eta\,\mathrm{d}V+\int_{\partial D}\frac{\delta^{\vee}F}{\delta\eta}(\eta,\mu)\cdot\delta\eta|\partial D \quad (3.4)$$

for all variations $\delta\eta$ and

$$\mathrm{D}_\mu F(\eta,\mu)\cdot\delta\mu=\int_D\delta\mu\cdot\frac{\delta^{\wedge}F}{\delta\mu}(\eta,\mu)\,\mathrm{d}V+\int_{\partial D}\delta\mu|\partial D\cdot\frac{\delta^{\vee}F}{\delta\eta}(\eta,\mu) \quad (3.5)$$

for all variations $\delta\mu$. Eqs. (3.4) and (3.5) do not uniquely determine the components of the functional derivatives. By applying the divergence theorem to these equations, one sees that we are free to add the gradient of a harmonic function (regarded as a density, etc., as is appropriate) to the interior term $\delta^{\wedge}F/\delta\eta$ (or $\delta^{\wedge}F/\delta\mu$) and subtract the corresponding normal component from the boundary term $\delta^{\vee}F/\delta\eta$ (or $\delta^{\vee}F/\delta\mu$) without changing the validity of (3.4) and (3.5). These partial functional derivatives can be uniquely specified by specifying Dirichlet boundary conditions for the harmonic function. We remain flexible about the choice as two different ones will be needed later.

Our definition of the bracket is motivated as follows. Let $\delta_{\partial D}$ be the Dirac delta measure on D which is concentrated on ∂D; dropping the densities from the boundary terms, define

$$\frac{\delta F}{\delta\eta}=\frac{\delta^{\wedge}F}{\delta\eta}+\delta_{\partial D}\frac{\delta^{\vee}F}{\delta\eta}\quad\text{and}\quad\frac{\delta F}{\delta\mu}=\frac{\delta^{\wedge}F}{\delta\mu}+\delta_{\partial D}\frac{\delta^{\vee}F}{\delta\mu}. \quad (3.6)$$

Using the functional derivatives defined above, we formally define the canonical Poisson bracket on $\mathrm{T}^*\mathscr{C}$ in the standard way,

$$\{F,G\}=\int_D\left(\frac{\delta F}{\delta\eta}\cdot\frac{\delta G}{\delta\mu}-\frac{\delta G}{\delta\eta}\cdot\frac{\delta F}{\delta\mu}\right)\mathrm{d}V. \quad (3.7)$$

If (3.7) is to be well-defined, we must avoid squares of delta functions and uniquely specify the functional derivatives. We can do both of these things by restricting our attention to one of the following two classes of functions: i) F such that $\delta^{\vee}F/\delta\eta=0$ or ii) F such that $\delta^{\vee}F/\delta\mu=0$. (In our derivation of the bracket on $\mathscr{N}$ we shall work with the second class.) If one wishes to have a larger class of admissible functions, another approach is possible. After making a choice of boundary conditions that makes the functional derivatives unique, we require that functionals F,G are such that

$$\frac{\delta^{\vee}F}{\delta\eta}\cdot\frac{\delta^{\vee}G}{\delta\mu}-\frac{\delta^{\vee}G}{\delta\eta}\cdot\frac{\delta^{\vee}F}{\delta\mu}=0. \quad (3.8)$$

In either case, substituting (3.6) into (3.7) gives the well-defined expression

$$\{F,G\}=\int_D\left(\frac{\delta^{\wedge}F}{\delta\eta}\cdot\frac{\delta^{\wedge}G}{\delta\mu}-\frac{\delta^{\wedge}G}{\delta\eta}\cdot\frac{\delta^{\wedge}F}{\delta\mu}\right)\mathrm{d}V+\int_{\partial D}\left(\frac{\delta^{\wedge}F}{\delta\eta}\bigg|_{\partial D}\cdot\frac{\delta^{\vee}G}{\delta\mu}+\frac{\delta^{\vee}F}{\delta\eta}\cdot\frac{\delta^{\wedge}G}{\delta\mu}\bigg|_{\partial D}\right)-\int_{\partial D}\left(\frac{\delta^{\wedge}G}{\delta\eta}\bigg|_{\partial D}\cdot\frac{\delta^{\vee}F}{\delta\mu}+\frac{\delta^{\vee}G}{\delta\eta}\cdot\frac{\delta^{\wedge}F}{\delta\mu}\bigg|_{\partial D}\right). \quad (3.9)$$

Now we map $\mathrm{T}^*\mathscr{C}$ onto $\mathscr{N}$ by the map

$\Pi_{\mathscr{N}}:\mathrm{T}^*\mathscr{C}\to\mathscr{N}$

which takes (η,μ) to $(\boldsymbol{v},\Sigma)$, where

$\Sigma=\partial(\eta(D))$

and

$\langle\boldsymbol{v}(x),\boldsymbol{w}(x)\rangle=\mu(X)\cdot\boldsymbol{w}(x)$

for all vector fields $w(x)$ on $\mathbf{R}^n$, with $x=\eta(X)$ and $\langle\,,\rangle$ the Euclidean inner product. The map $\Pi_{\mathscr{N}}$ is invariant under the right action of $G=\mathrm{Diff}_{\mathrm{vol}}(D)$, the group of volume preserving diffeomorphisms of D, and so induces a bijection

$$\overline{\Pi}_{\mathscr{N}}\colon G\backslash \mathrm{T}^*\mathscr{C}\to\mathscr{N}$$

which, given the correct topologies, is a diffeomorphism. Thus, by the theory of reduction of Poisson manifolds (see Marsden et al. [7] for a review), $\mathscr{N}$ inherits a Poisson structure. Explicitly, the brackets on $\mathrm{T}^*\mathscr{C}$ and $\mathscr{N}$ are related by

$$\{F,G\}\circ\Pi_{\mathscr{N}}=\{F\circ\Pi_{\mathscr{N}},G\circ\Pi_{\mathscr{N}}\}_{\mathrm{T}^*\mathscr{C}}. \qquad (3.10)$$

Given $F\colon\mathscr{N}\to\mathbf{R}$ possessing functional derivatives, let $\overline{F}=F\circ\Pi_{\mathscr{N}}$. Then a straightforward application of the chain rule gives

$$\mathrm{D}_\eta\overline{F}(\eta,\mu)\cdot\delta\eta=\mathrm{D}_\Sigma F(v,\Sigma)\cdot\langle\delta\eta,\boldsymbol{\nu}\rangle-\mathrm{D}_v F(v,\Sigma)\cdot(\delta\eta\cdot\nabla)v.$$

(Note that v regarded as one form is $(\mu\circ\eta^{-1})$, so an η variation causes a v variation – the evaluation points x or X are suppressed for clarity.) Also,

$$\mathrm{D}_\mu\overline{F}(\eta,\mu)\cdot\delta\mu=\mathrm{D}_v F(v,\Sigma)\cdot\delta v.$$

Thus, choosing the boundary condition $\delta^{\check{}}\overline{F}/\delta\mu=0$, we get

$$\frac{\delta^{\hat{}}\overline{F}}{\delta\eta}=-(\nabla v)\cdot\frac{\delta F}{\delta v},\quad \frac{\delta^{\check{}}\overline{F}}{\delta\eta}=\frac{\delta F}{\delta\Sigma}\boldsymbol{\nu}\,\mathrm{d}A \qquad (3.11)$$

and

$$\frac{\delta^{\hat{}}\overline{F}}{\delta\mu}=\frac{\delta F}{\delta v},\quad \frac{\delta^{\check{}}\overline{F}}{\delta\mu}=0, \qquad (3.12)$$

where one forms and vector fields have now been identified using the Euclidean metric. Here $(\nabla v)\cdot\delta F/\delta v$ is defined by contracting the v and $\delta F/\delta v$ indices; i.e. for any vector u, $\langle u,(\nabla v)\cdot\delta F/\delta v\rangle=\langle(u\cdot\nabla)v,\delta F/\delta v\rangle$.

Substitution of (3.11) and (3.12) into (3.9) yields (2.6). This then derives (2.6) and proves proposition 2.2.

Remark. The general principles of reduction show that the motion in Lagrangian representation can be reconstructed from that in Eulerian representation (and, of course, the motion in Lagrangian representation covers that in Eulerian representation consistent with the respective Poisson structures). Explicitly, given a solution $v_t(x)=v(x,t)$, $\Sigma(t)$, we construct $\eta_t(X)=\eta(X,t)$ by integrating the ordinary differential equation on $\mathscr{C}$ given by

$$\frac{\mathrm{d}\eta_t}{\mathrm{d}t}=v_t\circ\eta_t^{-1}$$

and then let $\mu_t=v_t\circ\eta_t^{-1}$, regarded as a one form over η_t. Then (η_t,μ_t) is an integral curve of the corresponding canonical Hamiltonian system on $\mathrm{T}^*\mathscr{C}$.

4. The second representation and reduced principal bundles

The variables for the second representation are (w,ϕ,Σ). Recall that the space of these triples (w,ϕ,Σ) is denoted by $\mathscr{N}'$. Here w is divergence free and tangent to Σ, so we must impose the boundary condition

$$\langle w,\boldsymbol{\nu}\rangle=0 \qquad (4.1)$$

and so variations are also constrained. The constraint on the variations may be obtained by differentiating a curve $w_s=\eta_{s*}\xi_s$, where ξ_s is divergence free on D and parallel to ∂D. We find that δw has the form

$$\delta w=w'+[w,u], \qquad (4.2)$$

where w' is divergence free and parallel to Σ, u is a divergence free vector field on D_Σ satisfying $\langle u,\boldsymbol{\nu}\rangle=\delta\Sigma$ and $[w,u]$ is the Lie bracket of vector fields on D_Σ.

Notice that w and ϕ are decoupled, but that w and Σ are now coupled through (4.1) and (4.2) and note that δw need not be parallel to the boundary.

Given variations δv and $\delta\Sigma$ we get the corresponding variations δw, $\delta\phi$, and $\delta\Sigma$ by letting $\delta v - [w, u]$ be decomposed into w' and $\nabla\delta\phi$, and, in (4.2), letting $u = \nabla N(\delta\Sigma)$ where N is the linear operator that takes functions on Σ with zero integral to functions on D_Σ modulo additive constants defined by

$$\Phi = N(f) \quad \text{satisfies} \quad \nabla^2\Phi = 0, \quad \frac{\partial\Phi}{\partial\nu} = f. \qquad (4.3)$$

This produces an isomorphism between the variations $(\delta v, \delta\Sigma)$ and $(\delta w, \delta\phi, \delta\Sigma)$.

Given a function $F(w, \phi, \Sigma)$ define $\delta F/\delta w$, $\delta F/\delta\phi$ and $\delta F/\delta\Sigma$ to be, respectively, a divergence free vector field parallel to Σ, a function on Σ with zero integral and a function on Σ modulo additive constants such that the differential of F on allowed variations (i.e. a tangent vector to $\mathcal{N}'$) satisfies

$$DF(w, \phi, \Sigma)\cdot(\delta w, \delta\phi, \delta\Sigma) = \int_{D_\Sigma}\left\langle \frac{\delta F}{\delta w}, \delta w\right\rangle \mathrm{d}^3x + \int_\Sigma\left(\frac{\delta F}{\delta\phi}\delta\phi + \frac{\delta F}{\delta\Sigma}\delta\Sigma\right)\mathrm{d}A. \qquad (4.4)$$

Remark. Note that δw, $\delta\phi$ and $\delta\Sigma$ are linked by (4.2), so, for example, $\delta F/\delta w$ is not simply given by differentiating F with respect to w holding ϕ and Σ fixed. However, one can relate the functional derivatives by changing variables using the constraints on the variations. Also, $\delta F/\delta\Sigma$ here is not the same as $\delta F/\delta\Sigma$ holding v fixed. To avoid confusion for the comparison, we temporarily denote $\delta F/\delta\Sigma$ in $\mathcal{N}'$ by $\delta' F/\delta\Sigma$. One gets

$$\left.\begin{aligned} \frac{\delta F}{\delta w} &= P\left(\frac{\delta F}{\delta v}\right), \\ \frac{\delta F}{\delta\phi} &= \left\langle \frac{\delta F}{\delta v}, \nu\right\rangle, \\ \frac{\delta' F}{\delta\Sigma} &= \frac{\delta F}{\delta\Sigma} - \left\langle w, \nabla N\left(\frac{\delta F}{\delta\phi}\right)\right\rangle, \end{aligned}\right\} \qquad (4.5)$$

where P indicates projection onto the component of the Weyl–Hodge decomposition parallel to Σ, by noting that we must have

$$\int_{D_\Sigma}\left\langle \frac{\delta F}{\delta v}, \delta v\right\rangle \mathrm{d}^3x + \int_\Sigma \frac{\delta F}{\delta\Sigma}\delta\Sigma\,\mathrm{d}A = \int_{D_\Sigma}\left\langle \frac{\delta F}{\delta w}, \delta w\right\rangle \mathrm{d}^3x + \int_\Sigma\left(\frac{\delta' F}{\delta\Sigma}\delta\Sigma + \frac{\delta F}{\delta\phi}\delta\phi\right)\mathrm{d}A$$

and decomposing δv into δw and $\delta\phi$ as specified earlier. (This calculation makes use of the vector identity

$$\int_{D_\Sigma}\langle\nabla f, [w, \nabla g]\rangle\,\mathrm{d}^3x = -\int_\Sigma\langle w, \nabla f\rangle\frac{\partial g}{\partial\nu}\,\mathrm{d}A,$$

where f and g are harmonic functions on D_Σ and w is a divergence free vector field on D_Σ parallel to Σ.)

The bracket on $\mathcal{N}'$ is given in the following:

Definition 4.1. For F, G: $\mathcal{N}' \to \mathbf{R}$ which possess functional derivatives, let

$$\begin{aligned} \{F, G\} = &\int_{D_\Sigma}\left\langle \omega, \left[\frac{\delta F}{\delta w} + \nabla N\left(\frac{\delta F}{\delta\phi}\right)\right] \times \left[\frac{\delta G}{\delta w} + \nabla N\left(\frac{\delta G}{\delta\phi}\right)\right]\right\rangle \mathrm{d}^3x \\ &+ \int_\Sigma\left(\frac{\delta F}{\delta\Sigma}\frac{\delta G}{\delta\phi} - \frac{\delta G}{\delta\Sigma}\frac{\delta F}{\delta\phi}\right)\mathrm{d}A \\ &+ \int_\Sigma\left\langle w, \nu \times \left[\nabla N\left(\frac{\delta F}{\delta\phi}\right) \times \nabla N\left(\frac{\delta G}{\delta\phi}\right)\right]\right\rangle \mathrm{d}A. \end{aligned} \qquad (4.6)$$

There are three ways to derive this bracket:

(a) directly from (2.6) by changing variables using (4.5),
(b) by repeating the reduction procedure in section 3 using (w, ϕ, Σ) in place of (v, Σ),
(c) by using the general formula for brackets on reduced principal bundles.

We shall omit the derivations using (a) and (b) (although it was (b) which first obtained the correct answer) and instead turn to method (c).

The key point which relates the present situation to bundles is that the material configuration space $\mathscr{C}$ described in section 3 may be considered as a principal bundle over $\mathscr{B}$, the manifold of surfaces Σ in $\mathbf{R}^3$ which are diffeomorphic to the boundary of the reference manifold D and bound a region D_Σ of volume equal to that of D. The structure group of this bundle is $G = \mathrm{Diff}_{\mathrm{vol}}(D)$, the group of volume preserving diffeomorphisms of D. The projection π from $\mathscr{C}$ to $\mathscr{B}$ takes η to $\Sigma = \partial(\eta(D))$. We endow $\mathscr{C}$ with the connection determined by choosing the horizontal subspaces to be $H_\eta = \{\nabla f \circ \eta | f$ is a harmonic function on $\eta(D)\}$. This defines for each $\eta \in \mathscr{C}$ the horizontal lift

$$h_\eta\colon \mathrm{T}_{\pi(\eta)}\mathscr{B} \to \mathrm{T}_\eta\mathscr{C}$$
$$\delta\Sigma \mapsto \nabla N(\delta\Sigma) \circ \eta. \tag{4.7}$$

Physically, $h_\eta(\delta\Sigma)$ may be thought of as the velocity field of the irrotational flow determined by the boundary motion $\delta\Sigma$.

We consider the "Sternberg Space" $\mathscr{C}^\# \underset{G}{\times} \mathscr{X}_\parallel^\# \cong \mathscr{N}'$, where $\mathscr{C}^\# = \{(\eta, \phi) | \eta \in \mathscr{C}$ and ϕ is a function on $\Sigma = \pi(\eta)$ with $\int_\Sigma \phi \, \mathrm{d}A = 0\}$, $\mathscr{X}_\parallel^*$ is the dual of the Lie algebra of G, elements of which are represented here as divergence free vector fields on D parallel to ∂D, and $\mathscr{C}^\# \underset{G}{\times} \mathscr{X}_\parallel^*$ is the quotient of $\mathscr{C}^\# \times \mathscr{X}_\parallel^*$ by the diagonal G action:

$$\psi \cdot ((\eta, \phi), \gamma) = ((\eta \circ \psi, \phi), P(\psi^{*\mathrm{T}}\gamma)).$$

where $\psi^{*\mathrm{T}}$ denotes the coadjoint action, given in our representation of $\mathscr{X}_\parallel^*$ by $\psi^{*\mathrm{T}}\gamma = (\mathrm{T}\psi^{-1})^{\mathrm{T}} \cdot \gamma \cdot \psi^{-1}$, where $^{\mathrm{T}}$ denotes transpose and $\mathrm{T}\psi$ is the tangent of ψ. (Note that $\mathscr{N}'$ is written here as triples in the order (Σ, ϕ, w).) We remark that $\mathscr{C}^\#$ is the pullback bundle which makes the diagram

$$\begin{array}{ccc} \mathscr{C}^\# & \to & \mathscr{C} \\ \downarrow & & \downarrow \\ \mathrm{T}^*\mathscr{B} & \to & \mathscr{B} \end{array} \tag{4.8}$$

commute. The horizontal lift h from $\mathscr{B}$ to $\mathscr{C}$ defines a connection one form Γ on $\mathscr{C}$, a horizontal lift $\tilde{h}$ from $\mathrm{T}^*\mathscr{B}$ to $\mathscr{N}'$ and a connection one form $\tilde{\Gamma}$ on $\mathscr{N}'$; we define the covariant differential of a function F on $\mathscr{N}'$ as follows:

$$\begin{aligned} &\mathrm{D}_\Gamma F(\Sigma, \phi, w) \cdot (\delta\Sigma, \delta\phi) \\ &\quad = \mathrm{D}F(\Sigma, \phi, w) \cdot \tilde{h}_{(\Sigma, \phi, w)}(\delta\Sigma, \delta\phi) \\ &\quad = \int_\Sigma \left(\frac{\delta F}{\delta \Sigma} \delta\Sigma + \frac{\delta F}{\delta \phi} \delta\phi \right) \mathrm{d}A + \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta w}, \delta w \right\rangle \mathrm{d}V, \end{aligned} \tag{4.9}$$

where $(\delta\Sigma, \delta\phi, \delta w) = \tilde{h}_{(\Sigma, \phi, w)}(\delta\Sigma, \delta\phi)$.

We calculate the third component, δw, of $\tilde{h}_{(\Sigma, \phi, w)}$ as follows. Let $(\eta_\varepsilon, \phi_\varepsilon, \gamma)$ be a curve in $\mathscr{C}^\# \times \mathscr{X}_\parallel^*$ tangent to the vector $(\boldsymbol{\delta\eta}, \delta\phi, 0)$. The projection of $\mathscr{C}^\# \times \mathscr{X}_\parallel^*$ onto $\mathscr{N}' = \mathscr{C}^\# \underset{G}{\times} \mathscr{X}_\parallel^*$ takes $(\eta_\varepsilon, \phi_\varepsilon, \gamma)$ to $(\Sigma_\varepsilon, \phi_\varepsilon, P_\varepsilon(\boldsymbol{u}_\varepsilon))$, where $\Sigma_\varepsilon = \partial(\eta_\varepsilon(D))$, P_ε denotes projection onto the parallel component of the Hodge decomposition with respect to Σ_ε and $\boldsymbol{u}_\varepsilon = \eta_\varepsilon^{*\mathrm{T}}\boldsymbol{\gamma}$. We assume, without loss of generality, that $\eta_0 = \mathrm{Id}$ and note that in the following calculation the symbol δ represents differentiation with respect to ε evaluated at $\varepsilon = 0$ (e.g. $\delta\boldsymbol{u} = \mathrm{d}/\mathrm{d}\varepsilon|_{\varepsilon=0}\boldsymbol{u}_\varepsilon$). A computation shows that

$$\delta\boldsymbol{u} = \boldsymbol{\delta\eta}[\boldsymbol{u}], \tag{4.10}$$

where $\boldsymbol{\delta\eta}[\boldsymbol{u}]^j = -(\delta\eta^i_{,j} u^i + u^j_{,i}\delta\eta^i)$. Since $w_\varepsilon = P_\varepsilon(\boldsymbol{u}_\varepsilon) = \boldsymbol{u}_\varepsilon - \nabla\alpha_\varepsilon$, for some harmonic functions α_ε, we find that $\delta w = \delta\boldsymbol{u} - \nabla\delta\alpha$. Also, $\boldsymbol{\delta\eta}[\boldsymbol{u}] = \boldsymbol{\delta\eta}[w]$, since $\boldsymbol{u} = \boldsymbol{u}_\varepsilon = w = \gamma$ at $\varepsilon = 0$. Now, using the horizontal lift from $\mathscr{B}$ to $\mathscr{C}$, we set $\boldsymbol{\delta\eta} = h_\eta(\delta\Sigma) = \nabla N(\delta\Sigma)$ and compute that

$$\delta w = \nabla N(\delta\Sigma)[w] + \text{a gradient}, \tag{4.11}$$

using the fact that

$$\nabla N(\delta\Sigma)[\nabla\alpha] = \nabla(\langle \nabla N(\delta\Sigma), \nabla\alpha \rangle).$$

(The gradient term is chosen to make the boundary condition for δw the same as the condition (4.2).) It follows that

$$\begin{aligned} &\mathrm{D}_\Gamma F(\Sigma, \phi, w)(\delta\Sigma, \delta\phi) \\ &\quad = \int_\Sigma \left(\frac{\delta F}{\delta \Sigma} \delta\Sigma + \frac{\delta F}{\delta \phi} \delta\phi \right) \mathrm{d}A \\ &\qquad + \int_{D_\Sigma} \left\langle \frac{\delta F}{\delta w}, \nabla N(\delta\Sigma)[w] \right\rangle \mathrm{d}V. \end{aligned} \tag{4.12}$$

According to the general bundle formula given in Montgomery, Marsden and Ratiu [8], the bracket on $\mathscr{N}'$ is

$$\{F,G\}(\Sigma,\phi,w) = \int_{D_\Sigma}\left(\left\langle -w, \left[\frac{\delta F}{\delta w},\frac{\delta G}{\delta w}\right]\right\rangle + \langle w, \Omega_{\Sigma,\phi}(\mathrm{J}\mathrm{D}_\Gamma F, \mathrm{J}\mathrm{D}_\Gamma G)\rangle\right)\mathrm{d}V + \{\mathrm{D}_\Gamma F, \mathrm{D}_\Gamma G\}_{\mathrm{T}^*\mathscr{B}}, \tag{4.13}$$

where the bracket in the first term denotes the usual Lie bracket on fields on D_Σ (minus the left Lie algebra bracket of Diff), $\Omega_{\Sigma,\phi}$ is the curvature of the connection $\tilde{\Gamma}$ on $\mathscr{N}'$, thought of as a bundle valued two form on $\mathrm{T}^*\mathscr{B}$, $\mathbf{J}$ is the mapping induced by the symplectic form on $\mathrm{T}^*\mathscr{B}$ which takes $\mathrm{T}^*(\mathrm{T}^*\mathscr{B})$ to $\mathrm{T}(\mathrm{T}^*\mathscr{B})$ and the bracket in the final term is the canonical bracket on $\mathrm{T}^*\mathscr{B}$ with partial derivatives replaced by covariant derivatives. Substituting the expression for $\mathrm{D}_\Gamma F$ given above, one finds

A straightforward (although somewhat lengthy) computation shows that this bracket is equal to the bracket given in (4.5).

Remarks. 1) In the computation of the curvature and canonical bracket terms, it is never necessary to explicitly determine a functional derivative corresponding to the Σ-component of $\mathrm{D}_\Gamma F$. In the curvature calculation, $\mathbf{J}\mathrm{D}_\Gamma F$ is projected via the canonical cotangent bundle projection onto $\delta F/\delta\phi \in \mathrm{T}\mathscr{B}$; the Σ-component does not enter into the calculation. In the canonical term we express the bracket in the form

$$\{F,G\}_{\mathrm{T}^*\mathscr{B}} = \int_\Sigma\left(\mathrm{D}_\Sigma F\cdot\frac{\delta G}{\delta\phi} - \mathrm{D}_\Sigma G\cdot\frac{\delta F}{\delta\phi}\right)\mathrm{d}A, \tag{4.15}$$

avoiding the need for a covariant functional derivative with respect to Σ.

2) The calculation of $\Omega_{\Sigma,\phi}$ involves a projection onto the parallel component of the Weyl–Hodge decomposition which need not be explicitly computed, since pairing the curvature field with the parallel field w annihilates the gradient component.

$$\{F,G\} = \int_{D_\Sigma}\left(-\left\langle w,\left[\frac{\delta F}{\delta w},\frac{\delta G}{\delta w}\right]\right\rangle - \left\langle w,\left[\nabla N\left(\frac{\delta F}{\delta\phi}\right),\nabla N\left(\frac{\delta G}{\delta\phi}\right)\right]\right\rangle + \left\langle\frac{\delta F}{\delta w},\nabla N\left(\frac{\delta G}{\delta\phi}\right)[w]\right\rangle - \left\langle\frac{\delta G}{\delta w},\nabla N\left(\frac{\delta F}{\delta\phi}\right)[w]\right\rangle\right)\mathrm{d}V + \int_\Sigma\left(\frac{\delta F}{\delta\Sigma}\frac{\delta G}{\delta\phi} - \frac{\delta G}{\delta\Sigma}\frac{\delta F}{\delta\phi}\right)\mathrm{d}A. \tag{4.14}$$

5. Generalized brackets and vorticity Casimirs

We now introduce a more general Poisson bracket which, while more complicated than the brackets already given, has the advantage that it admits a larger class of functions. In particular, functions of the form $\int_{D_\Sigma}\Phi(\omega)\,\mathrm{d}V$, where ω is the vorticity, do not possess functional derivatives of the form previously described but still will be shown to be Casimirs of the generalized bracket.

We consider first the case of an ideal fluid moving in a *fixed* region D in $\mathbf{R}^n$. The configuration space for such a fluid is $G = \mathrm{Diff}_{\mathrm{vol}}(D)$; we are concerned with the phase space T^*G reduced by the right action:

$G\backslash T^*G \approx \mathscr{X}_{\|}^*(D)$. (See Marsden and Weinstein [6].) We identify $\mathscr{X}_{\|}^*(D)$ and $\mathscr{X}_{\|}(D)$ via the L_2 pairing of elements $\boldsymbol{v}$ and $\boldsymbol{w}$ of $\mathscr{X}_{\|}(D)$,

$$\int_D \langle \boldsymbol{v}, \boldsymbol{w} \rangle \, \mathrm{d}V. \tag{5.1}$$

A function F: $\mathscr{X}_{\|}^*(D) \to \mathbf{R}$ is considered to have functional derivatives if for every element $\boldsymbol{v}$ of $\mathscr{X}_{\|}^*(D)$, there exist (i) $(\delta\hat{}F/\delta\boldsymbol{v})(\boldsymbol{v})$, an element of $\mathscr{X}_{\|}(D)$, and (ii) $(\delta\check{}F/\delta\boldsymbol{v})(\boldsymbol{v})$, a vector field on ∂D satisfying

$$\mathrm{D}F(\boldsymbol{v}) \cdot \delta\boldsymbol{v} = \int_D \left\langle \frac{\delta\hat{}F}{\delta\boldsymbol{v}}(\boldsymbol{v}), \delta\boldsymbol{v} \right\rangle \mathrm{d}V + \int_{\partial D} \left\langle \frac{\delta\check{}F}{\delta\boldsymbol{v}}(\boldsymbol{v}), \delta\boldsymbol{v}|\partial D \right\rangle \mathrm{d}A \tag{5.2}$$

for every variation $\delta\boldsymbol{v}$. (As in section 3, $\delta\hat{}F/\delta\boldsymbol{v}$ and $\delta\check{}F/\delta\boldsymbol{v}$ are not unique; the normal component of $\delta\check{}F/\delta\boldsymbol{v}$ can be modified arbitrarily using a harmonic gradient.) The Poisson bracket on $\mathscr{X}_{\|}^*(D)$ is computed, using an argument analogous to that given in section 3, to be

$$\begin{aligned}\{F,G\} = \int_D \left\langle \boldsymbol{\omega}, \frac{\delta\hat{}F}{\delta\boldsymbol{v}} \times \frac{\delta\hat{}G}{\delta\boldsymbol{v}} \right\rangle \mathrm{d}V + \int_{\partial D} \Bigg(\left\langle \boldsymbol{\omega}, \frac{\delta\hat{}F}{\delta\boldsymbol{v}} \times \frac{\delta\check{}G}{\delta\boldsymbol{v}} + \frac{\delta\check{}F}{\delta\boldsymbol{v}} \times \frac{\delta\hat{}G}{\delta\boldsymbol{v}} \right\rangle \\ + \left\langle \nabla(f_* - f), \frac{\delta\check{}G}{\delta\boldsymbol{v}} \right\rangle - \left\langle \nabla(g_* - g), \frac{\delta\check{}F}{\delta\boldsymbol{v}} \right\rangle \Bigg) \mathrm{d}A,\end{aligned} \tag{5.3}$$

where $\nabla f_* = (I-P)((\nabla\boldsymbol{v})\cdot\delta\hat{}F/\delta\boldsymbol{v})$ and $\nabla f = (I-P)(((\delta\hat{}F/\delta\boldsymbol{v})\cdot\nabla)\boldsymbol{v})$. For this to correspond to a well-defined bracket in Lagrangian representation, a restriction must be placed on the boundary terms to avoid squares of delta functions, just as in section 3. For example, if one of F or G should satisfy $\delta\check{}F/\delta v = 0$, then (5.3) is well defined.

In the two-dimensional case, the bracket (5.3) simplifies to

$$\{F,G\} = \int_D \left\langle \omega, \frac{\delta\hat{}F}{\delta v} \times \frac{\delta\hat{}G}{\delta v} \right\rangle \mathrm{d}V + \int_{\partial D} \left(\left\langle \nabla(f_* - f), \frac{\delta\check{}G}{\delta v} \right\rangle - \left\langle \nabla(g_* - g), \frac{\delta\check{}F}{\delta v} \right\rangle \right) \mathrm{d}A, \tag{5.4}$$

since, on ∂D, $\delta\hat{}F/\delta\boldsymbol{v}$ and $\delta\check{}G/\delta\boldsymbol{v}$ (likewise $\delta\hat{}G/\delta\boldsymbol{v}$ and $\delta\check{}F/\delta\boldsymbol{v}$) must be collinear and hence have trivial cross product.

We now restrict our attention to the two-dimensional case and show that a function of the form $C(\boldsymbol{v}) = \int_D \Phi(\omega)\,\mathrm{d}V$, where $\omega = \langle \boldsymbol{\omega}, \hat{z} \rangle$, $\hat{z}$ is the unit vector in the z-direction, and Φ is a C^2 function on D, is a Casimir. We will show that $\{C,G\} = 0$ for all functions G with functional derivatives such that $\delta\check{}G/\delta\boldsymbol{v} = 0$. A calculation shows that $C = \int_D \Phi(\omega)\,\mathrm{d}V$ has functional derivatives

$$\frac{\delta\hat{}C}{\delta\boldsymbol{v}} = P(\operatorname{curl}(\Phi'(\omega)\hat{z})) \quad \text{and} \quad \frac{\delta\check{}C}{\delta\boldsymbol{v}} = \Phi'(\omega)\hat{z} \times \boldsymbol{\nu}$$

(so here we choose $\delta^{\vee}C/\delta v$ to have zero normal component). Thus, using (5.4), and integration by parts,

$$\begin{aligned}\{C,G\} &= \int_D \left\langle \omega, P(\operatorname{curl}(\Phi'(\omega)\hat{z})) \times \frac{\delta^{\wedge}G}{\delta v} \right\rangle \mathrm{d}V - \int_{\partial D} \langle \nabla(g_* - g), \Phi'(\omega)\hat{z} \times \nu \rangle \,\mathrm{d}A \\ &= \int_D \left(\operatorname{div}\left(\Phi(\omega)\frac{\delta^{\wedge}G}{\delta v}\right) - \left\langle \frac{\delta^{\wedge}G}{\delta v}, ((I-P)(\operatorname{curl}(\Phi'(\omega)\hat{z}))\cdot\nabla)v \right\rangle \right. \\ &\quad - \left\langle (I-P)(\operatorname{curl}(\Phi'(\omega)\hat{z}), \left(\frac{\delta^{\wedge}G}{\delta v}\cdot\nabla\right)v \right\rangle \\ &\quad \left. + \left\langle (\nabla v)\cdot\frac{\delta^{\wedge}G}{\delta v} - \left(\frac{\delta^{\wedge}G}{\delta v}\cdot\nabla\right)v, (I-P)(\operatorname{curl}(\Phi'(\omega)\hat{z})) \right\rangle \right) \mathrm{d}V \\ &= 0.\end{aligned}$$

Thus C is a Casimir in the sense that $\{C,G\}=0$ for any G such that $\delta^{\vee}G/\delta v = 0$. In particular, $\{C,H\}=0$, where $H(v) = (1/2)\int_D |v|^2\,\mathrm{d}V$ is the standard Hamiltonian.

We note here that

$$\int_D \omega \left\{ \frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right\}_{xy} \mathrm{d}x\,\mathrm{d}y, \tag{5.5}$$

where $\{f,g\}_{xy} = (\partial f/\partial x)\partial g/\partial y - (\partial g/\partial x)\partial f/\partial y$ and $\partial f/\partial \omega$ is determined by the condition

$$\mathrm{D}F\cdot\delta\omega = \int_D \frac{\delta F}{\delta\omega}\delta\omega\,\mathrm{d}x\,\mathrm{d}y, \tag{5.6}$$

is *not* the appropriate bracket for two-dimensional fluid flow. The vorticity functionals defined above are *not* Casimirs for (5.5). In fact (5.5) differs from the correct bracket (5.4) by some non-trivial boundary terms. This can be seen if one computes $\{C,H\}$ using (5.5) alone; one gets a non-zero answer, which would contradict the conservation of vorticity by ideal fluid flow.

We now present the generalized bracket in the *free* boundary case, in two dimensions for simplicity. We say that a function F on $\mathcal{N}$ has functional derivatives if there exist

(i) $(\delta F/\delta\Sigma)(v,\Sigma)$ a function on Σ determined up to a constant,
(ii) $(\delta^{\wedge}F/\delta v)(v,\Sigma)$ a divergence free vector field on D_Σ, and
(iii) $(\delta^{\vee}F/\delta v)(v,\Sigma)$ a vector field on Σ

such that

$$\begin{aligned}\mathrm{D}F(v,\Sigma)\cdot(\delta v,\delta\Sigma) &= \int_{D\Sigma}\left\langle \frac{\delta^{\wedge}F}{\delta v}(v,\Sigma), \delta v\right\rangle \mathrm{d}V \\ &\quad + \int_\Sigma \left(\frac{\delta F}{\delta\Sigma}(v,\Sigma)\cdot\delta\Sigma + \left\langle \frac{\delta^{\vee}F}{\delta v}(v,\Sigma), \delta v\right\rangle\right)\mathrm{d}A\end{aligned} \tag{5.7}$$

for all variations $(\delta v, \delta\Sigma)$. (As usual, one can choose the normal component of $\delta^{\vee}F/\delta v$ at will.) The generalized bracket on $\mathcal{N}$ is

$$\{F,G\} = \int_{D_\Sigma} \left\langle \omega, \frac{\delta\hat{\ }F}{\delta v} \times \frac{\delta\hat{\ }G}{\delta v} \right\rangle \mathrm{d}V$$
$$+ \int_\Sigma \left(\left\langle \omega, \frac{\delta\hat{\ }F}{\delta v} \times \frac{\delta\check{\ }G}{\delta v} + \frac{\delta\check{\ }F}{\delta v} \times \frac{\delta\hat{\ }G}{\delta v} \right\rangle + \left\langle \frac{\delta F}{\delta \Sigma}\nu, \frac{\delta\hat{\ }G}{\delta v} \right\rangle \right.$$
$$\left. + \left\langle \nabla p_F, \frac{\delta\check{\ }G}{\delta v} \right\rangle - \left\langle \frac{\delta G}{\delta \Sigma}\nu, \frac{\delta\hat{\ }F}{\delta v} \right\rangle - \left\langle \nabla p_G, \frac{\delta\check{\ }F}{\delta v} \right\rangle \right) \mathrm{d}A, \tag{5.8}$$

where $\omega\hat{z} = \mathrm{curl}\ v$ and p_F is the solution of the Dirichlet problem: $\nabla^2 p_F = -\mathrm{div}\ ((\nabla v)\cdot\delta\hat{\ }F/\delta v)$ and $p_F|\Sigma = \delta F/\delta\Sigma - \langle(\nabla v)\cdot\delta\check{\ }F/\delta v, \nu\rangle$. The derivation of the bracket (5.8) is analogous to that given in section 3.

As in the fixed boundary case, functionals of the form $C(v,\Sigma) = \int_{D_\Sigma}\Phi(\omega)\mathrm{d}V$ are Casimirs of the generalized bracket in the sense that $\{C,G\} = 0$ for any function G on $\mathcal{N}$ with functional derivatives such that $\delta\check{\ }G/\delta v = 0$. The function C given above has the functional derivatives

$$\frac{\delta C}{\delta\Sigma} = \Phi(\omega), \qquad \frac{\delta\hat{\ }C}{\delta v} = \mathrm{curl}(\Phi'(\omega)z), \qquad \frac{\delta\check{\ }C}{\delta v} = \Phi'(\omega)\hat{z}\times\nu;$$

we omit the details of the calculation that $\{C,G\} = 0$, which involves only basic vector identities and repeated application of the divergence theorem.

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