Action Spectrum and Collisions in the Planar Three-body Problem.

– Richard Montgomery

Mathematics Dept. UCSC, Santa Cruz, CA 95064 USA
rmont@math.ucsc.edu

1. Introduction.

The action principle is returning to life in celestial mechanics. In our lectures at this conference Alain Chenciner and I reported how we used the direct method of the calculus of variations to find a surprising new orbit for the three-body problem with three equal masses, an orbit in which the masses chase each other around a fixed figure eight curve in the plane. (See the preprint [ChencMont] and the author's web page: http://www.orca.ucsc.edu/ rmont.) The paper[ChencVent], also of this year, is the only other paper I know of which in which the principle is used to obtain new results for the Newtonian N-body problem. Alain will report further on the use of this method elsewhere in this proceedings.

In this report I explain how and why collisions form the central obstacle to implementing the direct method of the calculus of variations. The two theorems presented say that if the infimum of the action over all closed collision-free paths representing a given free homotopy class is greater than or equal to a certain collision value then that infimum is realized by a collision orbit. At the end of the report I give a speculative picture of the action spectrum: the plot of action versus orbit type. I propose that the first, and perhaps the only possible interesting homotopy class as far as minimizing action is concerned is that of a figure eight in the shape sphere.

2. The direct method and the two ways compactness fails.

We review the direct method of the calculus of variations as it applies to the N-body problem. We begin by fixing a class Λ of paths in configuration space. The art of the method lies largely in constructing this class. Two standard choices of Λ are constructed by either fixing two submanifolds of configuration space, or by fixing a free homotopy class in the configuration space minus collisions. The first class consists of all paths which begin on one submanifold at time 0 and end on the other at time T. The second class consists of all collision-free closed curves which realize the given homotopy class in time T. In both cases when I say "all paths" I mean all absolutely continuous paths. Recall that a curve is "absolutely continuous" if it is differentiable a.e. and can be recovered by integraging its derivative. We then form the infimum of the action over Λ:

$$a(\Lambda) = inf_{c\in\Lambda} A(c).$$

where A is the standard action:

$$A = \int_c L dt,$$

with

$$L = \frac{1}{2}K + U$$

being the usual Lagrangian of mechanics. Thus the usual energy is

$$H = \frac{1}{2}K - U,$$

while $K = \Sigma m_a \dot{x}_a \cdot \dot{x}_a$ is twice the standard kinetic energy and $U = \Sigma m_a m_b / r_{ab}$ is the negative of the usual potential energy. Here we use the following notation. A typical path is written $c = (x_1(t), \dots, x_N(t))$ where x_a is the position of the ath body in a fixed inertial plane. Its tangent is the curve $\dot{c} = (\dot{x}_1, \dots, \dot{x}_N)$ of velocities. The mass of the ath body is m_a. And $r_{ab} = |x_a - x_b|$ is the distance between the ath and bth body.

By definition of infimum, there is a sequence $c_n \in \Lambda$ with $A(c_n) \to a(\Lambda)$. Such a sequence is called a *minimizing sequence*. The direct method of the calculus of variations proceeds by taking such a minimizing sequence $\{c_n\}$, showing that it converges to some curve c_*, showing that this c_* is sufficiently smooth, showing that c_* satisfies Newton's equations, and finally, showing that c_* has whatever additional properties one desires.

The only obstacles to completing this program are the two "noncompactnesses" of configuration space; Precisely, one should ask:

$$r_{ab}(c_n(t_n)) \to \infty? \qquad [NC1]$$

or

$$r_{ab}(c_n(t_n)) \to 0? \qquad [NC2]$$

In English, do some of the masses become infinitely separated from each other? Or do some of them collide? If we can exclude both possibilities, then we are guaranteed a solution with all the desired properties.

We explain. From now on we suppose that the center of mass is fixed at the origin:

$$\Sigma m_a x_a = 0.$$

Then the motion is bounded provided

$$r_{ab} < C,$$

for some positive constant C and all pairs a, b. Such a bound answers question [NC1], which is the easier of the two noncompactnesses. This bound can be achieved, for example by basing Λ on an interesting free homotopy class, one for which the length of any curve realizing it must tend to infinity as $r_{ab} \to \infty$. (These are the 'tied' classes of Gordon [Gordon].)

The set of paths whose action is less than or equal to a fixed constant is an equicontinuous family by an argument we will soon recall. The Arzela-Ascoli theorem asserts any bounded equicontinous sequence of curves in $\mathbb{R}^n$ (or indeed in any complete metric space) has a convergent subsequence. Thus the c_n converge in the C^0 sense to some continuous path c_*.

Here is the equicontinuity argument. Since U is positive, we have $\int_c \frac{1}{2}K \leq A(c)$. Now write

$$K = \langle \dot{c}, \dot{c} \rangle := \|\dot{c}\|^2$$

thus defining the usual mass inner-product on configuration space. Apply the Cauchy-Schwartz inequality $\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}$ to $f = \|\dot{c}\|$, $g = 1$ to obtain $\int \|\dot{c}\| \leq \sqrt{\int K}\sqrt{\int 1}$. Also use $\| \int \dot{c}\| \leq \int \|\dot{c}\|$. Integrating from time t to time s we conclude that

$$\|c(t) - c(s)\| \leq \sqrt{2A(c)}\sqrt{|t-s|}.$$

It follows that the set of curves with action less than a constant (e.g. less than $a(\Lambda) + 1$) are equicontinous.

Once we have c_*, it is rather standard functional analysis, involving the weak topology on the Sobolev space H^1 of paths with square integrable derivative, to show that:

- $A(c_*) = a(\Lambda)$ [P1]
- the set of collision times for c_* has measure zero [P2]
- on the complement of the collision times, c_* satisfies Newton's equations [P3]

See for example Gordon [Gordon], 965-967, for a proof. Thus, if we want an honest solution to Newton's equations, we are faced with the difficulty of showing that (NC2) does not occur, that is to say, that there are no collisions along c_*. If we cannot, then all we know about our alleged "solution" is that it is a concatenation of possibly countably many solutions, glued together continuously at the collision times. These strange piecewise "solutions" are called "generalized solutions" in much of the literature in this area [Ambrosetti], [Rabinowitz].

Open question: Suppose c_* has an isolated binary collision at time t_*. Does c_*, restricted to a small interval about t_*, correspond to a Levi-Civita regularized solution to Newton's equations?

There is no reason to believe the answer is yes. Of course, if it were yes, the whole notion of these generalized solutions would be more useful and palatable.

3. Gordon's work on Kepler.

In 1970 W. Gordon found out what happened when one applies the direct method to the Kepler problem. This work is remarkably prescient. He pointed out the main difficulties in applying the method, described some possible routes around them, and even described a 'toy problem' which turns out to include all the relevant features of the three body problem. We summarize his work.

Collision in the planar Kepler problem corresponds to passing through the origin. Consider the space Λ of all closed curves $c : [O, T] \to I\!R^2$ which avoid the origin, and whose winding number about the origin is nonzero.

Background Theorem. *[Gordon] The infimum over this class Λ is realized by any Keplerian orbit of period T, including the limiting case of the elliptic collision-ejection orbit which passes through the origin. Excluding the latter, these all have winding number 1 or -1.*

There are two especially important features here. The first is that all Keplerian orbits of the same period have the same action. Indeed, specifying the semi-major axis is the same as specifying the energy, or the action, or the period. If our Lagrangian is:

$$L = \frac{1}{2}m|\dot{x}|^2 + \gamma/|x|$$

then this action is given by:

$$A_{Kep} = A(m, \gamma; T) = \frac{3}{2}(4\pi^2 m\gamma^2)^{1/3}T^{1/3} \qquad [KEP].$$

The second feature is the concavity of the action as a function of the period T. By this we mean that whenever $T_1 + T_2 = T$ with T_1, T_2 positive then

$$A(T_1) + A(T_2) > A(T),$$

which follows simply from this same fact for the function $T^{1/3}$.

The importance of the second feature is that it eliminates multiple collision times. Gordon's proof is as follows. By the same steps as above, Gordon knew he had a minimizer satisfying properties [P1], [P2], [P3] above. Now if the minimizer did have multiple collision times, then these would separate it into countably many collison-free arcs with the sums of the time intervals of these arcs being the total period T. Along each arc the curve must be a solution, and hence an collision-ejection solution of the given duration. Now the concavity inequality shows that one decreases the action by replacing all these collision arcs by a single one of duration T.

Corollary. *Label the masses for the N-body problem in increasing order, so that m_1 is the smallest. (All masses are allowed to be equal, in which case any m_a will do.) Then the infimum of the action over all paths which suffer some collision in the time T is*

$$a(collision) = \frac{1}{2}\frac{3}{2}(4\pi^2)^{1/3}\frac{m_1 m_2}{(m_1 + m_2)^{1/3}}(2T)^{1/3}.$$

This infimum is realized by any sequence c_n in which masses 1 and 2 execute half of the collision-ejection orbit, while all other masses m_a remain fixed in time, far from 1 and 2 and far from each other, with their distances from 1 and 2 and from each other tending to infinity as $n \to \infty$.

Proof. One easily sees that the sequence defined in the statement of the corollary tends to this quantity, this being one-half of the action for the collision-ejection orbit of the 1-2 Kepler problem. To see that any other collision path has greater action, note that letting the distances $r_{1a}, r_{2b} \to \infty$ has the same effect as setting all the other masses $m_a = 0$, $a \neq 1, 2$ to zero in each term of K and U appearing in the action, and that this replacement decreases the action. Finally, to obtain the formula for $a(collision)$, note that the effective Lagrangian obtained upon dropping all these other masses is of the above Keplerian form, with $m = m_1 m_2/(m_1 + m_2)$ being the usual reduced mass, while $\gamma = m_1 m_2$. Thus $a(collision) = A_{Kep}(m, \gamma, T)$ for these choices of m and γ.

QED

Write

$$A_{Kep}(T) = \frac{3}{2}(4\pi^2)^{1/3} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} (T)^{1/3},$$

so that the estimate in the corollary is by $\frac{1}{2} A_{Kep}(2T)$. By an argument nearly identical to the one just given we have

Theorem. *[Kepler at infinity]. Label the masses for the N-body problem as above. Consider either of the following two problems. (A): Minimize over all periodic orbits having a collision. (B): Minimize over all periodic orbits in the free homotopy class defined by having 1 and 2 make one full turn about each other while the rest of the bodies remain at rest, far away. The infimum for either problem is A_{Kep}. The infimum is realized by a sequence c_n in which masses 1 and 2 execute a Keplerian two-body while all the other masses remain fixed over the time interval, and their distances from these two moving bodies, and from each other are of the order $O(n)$, so as to tend to infinity as $n \to \infty$. The 1-2 Kepler motion is a collision-ejection orbit for problem (A) and a 1-2 Keplerian orbit for problem (B).*

The theorem above was crucial in our work with Chenciner [ChencMont]. The value A_{Kep} was denoted A_2 there.

Here is a result regarding triple collisons.

Lemma. *For the planar 3 body problem the infimum of the action over all periodic paths of period T which have a triple collision is realized by doubling the Lagrange homethetic solution.*

Proof. We recall Saari's decomposition of the kinetic energy, which is to say the splitting into reduced (deformation) and rotational motion:

$$K = \dot{r}^2 + \frac{1}{4} r^2 \|\dot{\sigma}\|^2 + \frac{\|C\|^2}{r^2}.$$

Here

$$r^2 = I := \langle x, x \rangle$$

is the total moment of inertial of the triangle formed by the three bodies. The variable σ lies on the unit sphere, and $\|\dot{\sigma}\|^2$ is the length squared of the derivative of the moving unit vector $\sigma(t)$, which is to say that it corresponds to the Riemannian metric on the unit sphere. The factor of 1/4 is essential, and will be explained later on. The vector C is the angular momentum and the term containing it represents the rotational kinetic energy. Moreover the potential term is given by

$$U = \tilde{U}(\sigma)/r.$$

The homogenized potential $\tilde{U}$ is a function on the sphere of crucial import to the three-body problem. Its absolute minimum occurs at the Lagrange points $r_{12} = r_{23} = r_{31}$ and there we have

$$\tilde{U}_{Lag} = \frac{m_1 m_2 + m_2 m_3 + m_3 m_1}{m_1 + m_2 + m_3}.$$

It follows that for any motion we have

$$L \geq \frac{1}{2} \dot{r}^2 + \tilde{U}_{Lag}/r.$$

with equality if and only if the shape of the triangle is equilateral (i.e. Lagrange) and this triangle are not spinnning. The right hand side is a Kepler Lagrangian, as treated by Gordon. Its absolute minimum, over all periodic solutions having a collision (i.e. $r = 0$ at some time) is as given by Gordon, corresponding to the following Lagrange's homothetic solution in to triple collision and then back out.

QED

4. Homotopy Dreams and Shape Space.

The configuration space for the collision-free N-body problem has a rich fundamental group, the colored braid group on N strands. This rather complicated infinite group is generated by the ij -binaries: the loops in which masses i and j make one turn around each other while the other masses stay motionless, far away. The conjugacy classes of the fundamental group are, as always, in one-to-one correspondence with free homotopy classes.

HQ1: Is every free homotopy class realized by an action-minimizing periodic collision-free solution to the Newtonian N-body problem?

HQ2: If not, which classes are represented?

These questions were inspired in large part by conversations with Hsiang. See [Hsiang] for his work in this area. These are basic questions we are pursuing.

Cheating. In [Braids] we cheated by replacing the Newtonian potential function U by a so-called "strong-force" potential. With this change we were able to prove that the answer to HQ1 'yes' or nearly so. By a "strong force potential" we mean one of the form $U_{s.f} = \Sigma f_{ab}(r_{ab})$ where the two-particle potentials $f_{ab}(r)$ are positive smooth functions, tend to 0 as $r \to \infty$ and, most importantly satisfy $f(r) \geq C/r^2$ provided $r \leq \delta$, for some constants C, δ. For example, a good choice is $f_{ab}(r) = m_a m_b/r + \epsilon(r)/r^2$ where the non-negative bump function $\epsilon(r)$ is 0 if $r > 10^{-26}$ and 1 if $r < 10^{-27}$.

Lemma. *Any collision path for a strong force potential has infinite action.*

Proof. Suppose masses 1 and 2 collide at time 0. Write r for r_{12}, and only consider that part of the path for which $r < \delta$. The Lagrangian L along this part of the path is bounded below by $C(\dot{r}^2 + 1/r^2)$ for some constant C. But $\dot{r}^2 + 1/r^2 \geq |2\dot{r}/r| = 2|\frac{d(logr)}{dt}|$. Consequently $A(c) \geq C|\int d(logr)| = +\infty$, since $|logr|$ tends to infinity as $r \to 0$.

QED

Thus, for strong-force potentials we can completely ignore collisions within minimizing sequences. The only obstruction to getting solutions is insuring that r_{ab} is bounded, i.e. the noncompactness [NC1] above. This bound can be achieved by choosing appropriate free homotopy classes, those classes which Gordon called "tied" to the singularities. In [Braids] we proved that almost every class is so tied, and thus obtained the theorem. We also gave a homological characterization of "tied". The sense of "almost every" is as follows. A set P of integers is said to have density 1 if $\#(P \cap [0, N])/N$ tends to 1 as $N \to \infty$. The braid groups are finitely generated, so countable, and a similar notion of density makes sense. The set of tied classes have density 1 in this sense. We use "almost every" as a synonym for "of density 1" below.

Background Theorem. *[Braids]. Under the strong force assumption on the potential, almost every (as described above) free homotopy class is realized by a collision-free periodic solution to the N-body problem with this potential.*

Shape Space. We return to the planar three-body problem, and its shape space. The configuration space for the problem is $Q = \mathbb{C}^3$ for three copies of the plane. Write G for the group orientation preserving isometries of the Euclidean plane. The *shape space* is defined to be the quotient space: $C = Q/G$. To understand C, first fix the center of mass to be the origin, thus defining the linear subspace $Q_0 = \{x : \Sigma m_i x_i = 0\} \subset Q$. This is to be identified with $Q/translations$. The subgroup $S^1 \subset G$ of rotations about the origin keeps this center of mass fixed, and we have $C = Q_0/S^1$. Shape space inherits a metric from the kinetic energy metric on Q. The distance between two points of C is defined to be the kinetic energy distance between the corresponding G orbits in Q. This metric is Riemannian away from triple collision, and we have encountered it earlier. It is the Riemannian metric whose kinetic energy is the term $\dot{r}^2 + r^2\|\dot{\sigma}\|^2/4$ arising in Saari's decomposition of the kinetic energy above.

Q_0, with the restricted kinetic energy, is equal to $\mathbb{C}^2$ with its standard inner product, the real part of the Hermitian inner product. Thus

$$I = |z_1|^2 + |z_2|^2$$

where (z_1, z_2) are standard linear complex coordinates on $\mathbb{C}^2$, and can be formed by normalizing Jacobi's coordinates. The rotation group acts by scalar multiplication of the vector z by a unit complex number. Thus $C = \mathbb{C}^2/S^1$. This quotient is well-known to be Euclidean three-space, topologically. The quotient map $Q_0 \to Q_0/S^1 = C$ is the Hopf map, also called the Kuustanheimo-Steiffel map $\mathbb{C}^2 \to \mathbb{R}^3$.

C is not isometric to Euclidean space; rather it is isometric to the cone over the two-sphere S of radius 1/2. Here we use

Definition. *The cone $Cone(X)$ over a Riemannian manifold (X, d^2s_X) is formed by introducing an additional radial variable $r \in [0, \infty)$ and forming the Riemannian metric $dr^2 + r^2 d^2 s_X$ on on $X \times (0, \infty)$. The associated distance function extends continuosly to the topological cone $Cone(X) = (X \times [0, \infty))/ \sim$ where the "$\sim$" means that all points of the form $(x, 0)$ are to be identified with a single point, written 0, and called the "cone point" of $Cone(X)$. The parameter r is the distance from the cone point.*

This sphere S over which C is the cone represents the space of oriented similarity classes of triangles. We call it the *shape sphere*. This is the same sphere drawn and described in Moeckel's beautiful article [Moeckel]. The cone point of C is triple collision. The distance r from triple collision is the square root of the moment of ineria, as we saw earlier, the r arising in the kinetic energy splitting formula. The sphere has three marked points, the three binary collision points, which are the poles of U restricted to $S = \{r = 1\} \subset C$. These singularities are lie on the equator of the sphere. This equator represents the locus of all collinear triangles, or triangles of zero area. Correspondingly, the shape space minus collisions is homeomorphic to $\mathbb{R}^3$ minus three rays through the origin, and is homotopic to the shape sphere S minus three points.

Write $C^* = \mathbb{R}^3 \setminus \{$ three rays $\}$ for the collision-free reduced configuration space. Any closed curve within C^* can be homotoped so as to intersect the collinear plane transversally, and so as to not intersect any given sector twice in a row. The intersections with sectors represent eclipses for the corresponding motion. This shows that the free homotopy classes for the (reduced) planar three-body problem can be uniquely labelled by cyclic words in the eclipses. For example a 12 binary orbit represents the word $12 = 21$. And the figure eight orbit of [ChencMont] represents the word 123123. The grammatical rules for the words are: (1) an even number 2ℓ of letters. (2) No stutterring (no ii) , when the word is viewed as cyclic, i.e. among any of its cyclic permutations. (Thus 2132 is not allowed since it equals 1322 when viewed as being written out on a circle.)

Our question HQ1 for the planar three body problem thus amounts to asking: Can any periodic sequence of eclipses, subject to the above grammatical rules, be realized by a periodic orbit? Can this orbit be taken to minimize the action over this homotopy class?

5. Orbit Surgery and shrinking.

We recall the fundamental scaling symmetry for Newton's equations with the Newtonian $1/r$ potential:

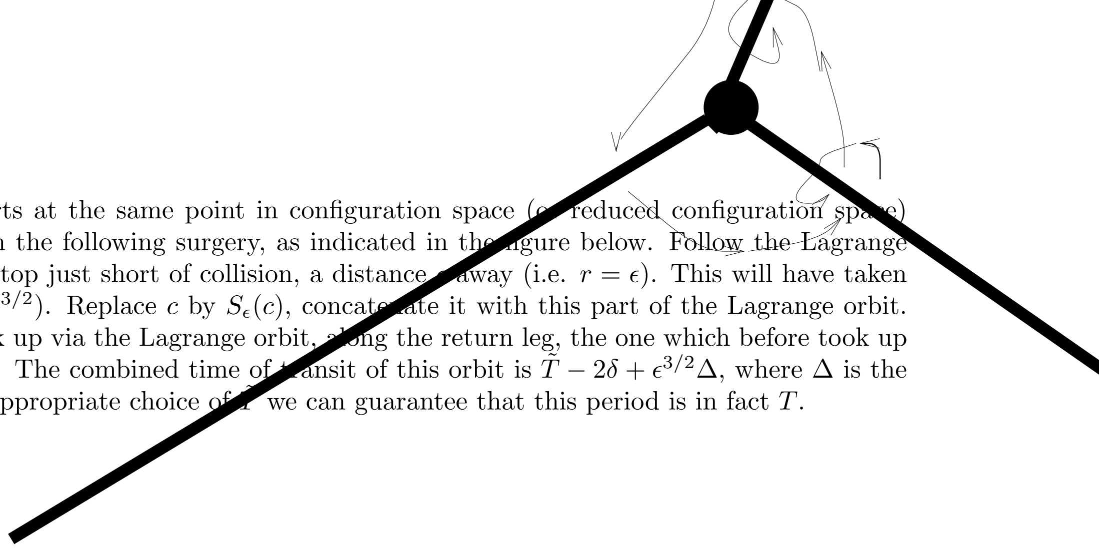
$$c(t) \mapsto S_\lambda(c)(t) = c_\lambda(t) := \lambda c(\lambda^{-3/2} t).$$

This scaling maps solutions to solutions, scaling the potential and kinetic energies in the same way so that

$$A(S_\lambda(c)) = \sqrt{\lambda} A(c).$$

If the curve c is periodic with period T then $S_\lambda(c)$ is periodic with period $\lambda^{3/2}T$. Note that S_λ is a deformation of the configuration space ($\mathbb{C}^2 \setminus \{collisions\}$) which preserves rays, and in particular takes collision-free paths to collision-free paths without changing their free homotopy type. Since S_λ commutes with rotations it induces a homotopy of the collision-free reduced configuration space C^* also, and we will use the same symbol for this dilation as well.

Let us fix such a free homotopy class α. Choose a representative c for this class which starts and ends at a Lagrange configuration. Fix a time T. For $\tilde{T}$ close to T, consider the period $\tilde{T}$ Lagrange collision-ejection solution, starting at the same configuration, up to scale. It reaches triple collision at time $\tilde{T}/2$. We may assume that our curve c starts at the same point in configuration space (or reduced configuration space) including scale. Now perform the following surgery, as indicated in the figure below. Follow the Lagrange orbit down to collision, and stop just short of collision, a distance ϵ away (i.e. $r = \epsilon$). This will have taken a time $\tilde{T}/2 - \delta$ with $\delta = O(\epsilon^{3/2})$. Replace c by $S_\epsilon(c)$, concatenate it with this part of the Lagrange orbit. To close the loop, return back up via the Lagrange orbit, along the return leg, the one which before took up the interval $\tilde{T}/2 + \delta \leq t \leq \tilde{T}$. The combined time of transit of this orbit is $\tilde{T} - 2\delta + \epsilon^{3/2}\Delta$, where Δ is the period of the original c. By appropriate choice of ϵ we can guarantee that this period is in fact T.



The action is that of the Lagrange path, minus a little bit for the final collision leg we skipped, plus $\sqrt{\epsilon}A(c)$. As $\epsilon \to 0$ this tends to the Lagrange action. The end result is a family of periodic orbits, all representing the homotopy class α, and with actions tending to the action A_{Lag} of the Lagrange homothetic orbit. We have proved:

Theorem. *Let $\Lambda(\alpha)$ denote the set of all collision free paths realizing the free homotopy class α in time T, and let $a(\alpha) = a(\Lambda(\alpha))$ be the infimum of the action over this set of loops, as in the beginning of this article. Then*

$$a(\alpha) \leq A_{Lag}$$

with equality if and only if there is a minimizing sequence which converges to the Lagrange homothetic collision-ejection path.

A Keplerian application

We apply this same idea to the Kepler problem. Write $A_{Kep}(T)$ for the action of a Keplerian orbit making one tour around the sun (origin). The reader will find a formula for this action above (eq. [KEP]). Let us write $a(k,T)$ for the infimum of the action over all closed curves which wind k times around the sun in time T. Thus $a(1,T) = A_{Kep}(T)$. The following lemma is a slight addition to Gordon's fundamental minimization theorem, quoted above.

Lemma. *For all $k \neq 0$ we have $a(k,T) = a(1,T)$. Moreover if $|k| > 1$ then a minimizing sequence c_n is realized by following the collision-ejection orbit almost into the sun, stopping a distance $1/n$ short of the sun, making k quick tours around the sun, and then returning along the same collision-ejection orbit.*

The idea of the proof is identical to the above, using the same scaling and concatenation trick. We leave the details to the reader.

6. The action spectrum

By the action spectrum we mean the value of the action of orbits (or infimums of actions) plotted against some way of indexing these orbits. We are interested in action minimizing periodic orbits in a given free homotopy class, so the index should be some kind of homotopy index. Thus we mark several points indexing either possible homotopy types or the two collision possibilities, and we have put in conjectural possibilities for the corresponding minimizing action $a(\alpha)$. Combining the theorem from section 5 and the theorem from section 3 we obtain

$$A_{Kep} \leq a(\alpha) \leq A_{Lag},$$

which is valid for any nontrivial free homotopy class α in the reduced collision free configuration space. (The inequality $a(\alpha) \leq A_{Lag}$ is also true in the full collision-free configuration space, and the other inequality probably holds as well.) Moreover equality at either end almost certainly means that this infimum is not realized. In other words, there is not an actual periodic orbit having the given action, but rather a sequence of paths tending to collision as in thee theorems, with action tending to the value in question.

Caveat: The inequality $A_{Kep} \leq A(c) \leq A_{Lag}$. does not hold for the figure eight curve constructed in [ChencMont]. Indeed there is no reason it should pertain, as that curve was constructed with the aid of imposing additional discrete symmetries, and does not achieve the infimum over its homotopy class, but rather over its homotopy class with these additional symmetries imposed.

The inequality $A_{Kep} \leq a(\alpha)$ deserves a bit of discussion. Being a nontrivial homotopy class, any curve which realizes α must make a tour around at least one of the collision rays. Say it tours the 12 ray. Now forget the third mass, thus replacing the action by the 12 Keplerian action. The infimum of this action over all loops making a tour around the 12 collision is as given by Gordon, hence the bound.

The big question is: are there any classes for which the infimum $a(\alpha)$ lies strictly between these two extremes? It seems the only viable candidate is a figure eight in shape space: e.g. the eclipse sequence 1232.

We show now that triple collisions will be avoided for the minimizer over this class, at least for nearly equal masses. The idea is the same as in [ChencMont]. We construct a test curve c which realizes this homotopy class, and whose action $A(c)$ is less than A_L. Since by the lemma above A_L is the infimum of the actions of all period T orbits suffering a triple collision, this inequality will exclude triple collisions. To construct the test curve c, assume that m_2 is the largest of the three masses. Then the maximum of the reduced potential $\tilde{U}$ among the three Euler configurations occurs at E_2, the one in which m_2 lies between m_1 and m_3. It follows that the equipotential curve $\tilde{U} = \tilde{U}(E_2)$ has the shape of an eight passing through E_2, and realizes the homotopy class 1232. Let $c(I)$ be this equipotential curve, on the sphere of radius I in the reduced configuration space C. As in [ChencMont], choose $I = I_*$ so that $A(c(I_*))$ is minimized among the $A(c(I))$. This $c = c(I_*)$ is our test curve. Its action $A(c)$ is computed as in [ChencMont] to be $A(c) = a(\ell; T) = \frac{3}{2}\tilde{U}(E_2)^{2/3}\ell^{2/3}T^{1/3}$ where ℓ is the length of the normalized curve $c(1)$ on the unit sphere. (The constant $\tilde{U}(E_2)$ plays the role of γ in our formula [Kep] and ℓ plays the role of m^2.) To obtain the desired inequality $A(c(I)) < A_L$ consider the case in which all three masses are equal. Then the curve $c(1)$ limits to two-thirds of the test curve $\tilde{U} = \tilde{U}_E$ used in [ChencMont]. (That full equipotential curve was a double covering of the equator passing through all three Euler points.) This 2/3 of a curve is not smooth, having corners at E_1 and E_3, but no matter: it is continuous and closed. We find that $\frac{\tilde{U}(E_2)=\tilde{U}_E=5}{\sqrt{2}}$. The condition $a(\ell; T) < A_L(T)$ is equivalent to the condition $\ell < \frac{6\sqrt{2}}{5}\pi$. This is indeed true, as $\ell = 8\ell_0$ where ℓ_0 is the length computed in [ChMont], and which is a bit less than $\pi/5$. (The quantity ℓ_0 was one-twelfth the length of the full test curve there.) Now, by continuity, the desired inequality remains in force for nearly equal masses.

Excluding binary collisions for this shape space eight is almost certainly a more difficult matter, and is by no means assured. Indeed, there is a candidate binary collision solution which is the limit of curves realizing the shape space eight and which may realize the infimum. This is Schubart's orbit [Schubart]. Does it realize the infimum? We do not know. This is a problem for the future.

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