RESOLVING SINGULARITIES WITH CARTAN'S PROLONGATION

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1. Introduction. History

Singularities, contact geometry, and blow-up frequently appear in Arnol'd's work. We are honored to have the opportunity here to present a drama in which these three actors play leading roles.

The standard method for resolving a plane curve singularity is blow-up. We will describe an alternative method which we call prolongation, in honor of Cartan's work in this direction. See [4], [5], and especially the last few sections of [3]. The prolongation we will be describing is known to many algebraic geometers as Nash blow-up. Our main result, Theorem 6.2 asserts that the two methods yield the same resolution for unibranched singularities.

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Both methods consist of transforming the given singular plane curve, and the space in which it lives. Iteratively applying the methods eventually yields a non-singular curve. In both cases, the points of the new space are marked lines: lines with points marked on them, and the marked points trace out the original (untransformed) curve. In blow-up the lines are the secant lines connecting the marked point to the singular point. The new space for the first blow-up is a surface, the blow-up of the plane at a point, and upon subsequent applications of blow-up the spaces continue to be surfaces. In prolongation, on the other hand, the lines are tangent lines to the curve at the marked point. The new space for the first prolongation is the space of all marked lines in the plane and forms a 3-dimensional contact manifold, the projectivized tangent bundle of the plane. The new curve in prolongation is tangent to the contact distribution. Upon each subsequent iteration of prolongation the dimension of the space increases by one, forming a $\mathbb{P}^1$-bundle over the previous space. These spaces are endowed with a rank 2 distribution and the prolonged curve is tangent to this distribution. The dimensional difference between the two spaces of blow-up and prolongation makes it somewhat unclear how to compare the resolved curves for the two cases. We solve this problem by comparing the resolution diagrams for the two methods, these diagrams being combinatorial skeletons of the resulting resolutions.

History. Kollar, [11] in Chapter 1 of his recent book describes at least 14 methods for resolving plane curve singularities, blow up being one. Kollar does not describe prolongation, but was well-aware of it [10] under the name "Nash blow-up". It can be found under that alternative name in the algebraic geometry text [9].

Prolongation, in the guise described here, has its roots in the works of E. Cartan [4], [5]. Robert Bryant explained it to one of us and used it in his paper [3]. We used it in this guise in [12] and [13]. Cartan's prolongations consisted of attaching new variables, curves, transformations, distributions, etc., which corresponded to derivatives of old objects. In our case, the primary objects are curves, and the prolonged objects consisting of the primary curve together with its moving tangent line, viewed as a moving point in an appropriate projective space.

Prolongation seems to have first appeared in algebraic geometry by way of a conversation between John Nash and Hironaka. See the introduction to [15] for the story. Consequently, this incarnation of prolongation is known as Nash blow-up in the Algebraic Geometry literature.

Start with a k-dimensional algebraic or analytic subvariety A in an n-dimensional affine space E^n. Let $Gr_k(n)$ denote the Grassmannian of k-planes in n-dimensions, and let $\Sigma \subset A$ be the singular locus of A. For $p \in A$, $p \notin \Sigma$, the tangent space T_pA is a point of $Gr_k(n)$ so that we can define the Gauss map $G : A \setminus \Sigma \to Gr_k(n)$, by $G(p) = T_pA$. The Nash blow-up (or prolongation) of A is the *closure* in $A \times Gr_k(n)$ of the graph of this Gauss map. We write $P(A)$ for this set. If we replace E^n by an n-dimensional smooth algebraic or analytic manifold S, so that $A \subset S$ is a singular analytic subvariety, then the Gauss map takes values in a bundle $B \to S$ whose fiber is $Gr_k(n)$. The same construction goes through, and $P(A) \subset B$. Assuming that $P(A)$ is again analytic or algebraic, the process can be iterated, to arrive at spaces $P^{j+1}(A) = P(P^j(A))$, $j = 2, 3, \ldots$ contained in spaces B_j.

In this paper we will use the word **prolongation** *rather than Nash blow-up.*

For general k and n the outcome $P^j(A)$ of this iterated prolongation is poorly understood. When $k = 1$ so that A a curve, and for general embedding dimension n, Nobile [14] has proved that for j sufficiently large, the jth prolongation $P^j(A)$ forms a smooth embedded curve. In the particular case $k = 1$ and $n = 2$ of planar curve singularities two of us reproduced Nobile's result independently in [13]. We also obtain much more information than can be found in Nobile, including a formula for the minimum j in terms of the Puiseux characteristic of the curve, and details of how the prolongation touches the fibers of the fibrations associated to passing from j to $j + 1$. This 'touching' information is encoded in the RVT code of the curve (as defined below) and is essential for forming the prolongation analogue of the exceptional divisors.

In this paper we continue to be entirely concerned with the case $k = 1, n = 2$ of a singular plane curve. We base our development on two properties of prolongation peculiar to this case. Neither of these properties seem to have been pointed out in the Nash blow-up literature.

Property A: The k-fold prolongation of a plane curve lies in a $k + 2$-dimensional space and is tangent to a distinguished 2-plane field, or rank 2 distribution, on that space. When $k = 1$ this manifold with its distribution is the 3-dimensional contact manifold of marked lines in the plane described above.

Property B: The exceptional fibers of blow-up have prolongation analogues which we call critical curves.

The combinatorics of intersections amongst the exceptional fibers and the proper transform are the end-product of resolution by blow-up. So, to compare resolution by prolongation with resolution by blow-up, we need the critical curves appearing in Property B. They are the prolongation analogues of the exceptional fibers. Perhaps the most difficult part of the present work was figuring out how to encode the combinatorics of the critical curves. The critical curves do not appear in the standard Nash blow-up.

There is a third property central to the case $k = 1, n = 2$ but which we will not make use of here.

Property C: The underlying symmetry group of prolongation is the group of contact transformations of the 3-dimensional contact manifold, which arises on the first prolongation.

Property C played the central role in [13]. There is a whole theory of resolving Legendrian curve singularities to be explored. In this paper, we have used the smaller group of surface diffeomorphisms, which is strictly contained in the contact diffeomorphisms, as our underlying symmetry group. This philosophical change of viewpoint, from contact symmetries to surface symmetries, accounts for many differences between the book [13] and the current paper.

2. Plane curves

Curves in the plane can be represented either as level sets $f(x, y) = 0$ or as the images of parameterizations: $t \mapsto (x(t), y(t))$. To pass from the first description to the second near a regular point use the implicit function theorem, and near a singular point p use the Newton-Puiseux expansion, an algorithm for expressing the curve locally as the finite union of images of a parameterized curves, called the branches of the singularities at p. See [2], [16], or [6], for details on the Newton-Puiseux expansion.

In this paper we will be concerned with germs of analytic unibranched singularities: singular curves consisting of a single branch. Examples are $x^p - y^q = 0$ where p and q are relatively prime integers. Such a curve is parameterized as $x = t^q, y = t^p$.

After a rotation of the (x, y) plane, any unibranched analytic singularity can be parameterized as

$$x = t^m, \quad y = \Sigma_{i>m} a_i t^i. \tag{2.1}$$

It is worth pointing out that being unibranched for an analytic curve germ singularity $f(x, y) = 0$ is equivalent to f being irreducible, in the usual sense of algebra, within the space of complex analytic function germs of two variables. See for example, Chapter 5, especially Corollary 5.1.8, of [7].

We work over the complex numbers $\mathbb{C}$ so that x, y and the parameter t all take values in $\mathbb{C}$. All of our definitions and constructions here carry through for real analytic plane curves but $\mathbb{C}$ is the traditional field over which to perform blow-up so we will work there.

The following notion will be useful.

Definition 2.1. A curve germ $t \mapsto c(t)$, $t \in \mathbb{C}$ defined near $t = 0$ is well-parameterized if there is a neighborhood of $t = 0$ such that the parameterization is one-to-one.

Example. The standard parameterization $x = t^2, y = t^3$ makes the the cusp $y^2 = x^3$ well -parameterized. But if we parameterize the cusp by $x = t^4, y = t^6$ then this parameterized curve is not-well parameterized.

Remark. If we work over the reals, a somewhat different definition of well-parameterized must be given. See [13].

Remark. We have the following number-theoretic way of establishing whether or not an analytic curve is well-parameterized. Suppose the curve to be given by equation (2.1). Let $supp(y) \subset \mathbb{N}$ be the set of exponents i occurring in the expansion of $y(t)$ such that $a_i \neq 0$. Then $c(t)$ is well-parameterized if and only if the greatest common divisor of $m \cap supp(y)$ is 1.

Henceforth, we will fix our primarily on singular well-parameterized planar curve germs.

3. Prolongation

Let c be a complex analytic curve in a smooth complex manifold M^n. Its singular points Σ are discrete. At each non-singular point $p \in c \backslash \Sigma$ the tangent line $T_p c$ to c is uniquely defined, and can be viewed as a point in the projectivized tangent bundle $\mathbb{P}TM$. The closure of the set of points $(p, T_p C), p \in c \setminus \Sigma$ is defined to be the first prolongation of c, and is denoted by c^1. Away from Σ, the projection $\mathbb{P}TM \to M$ maps c^1 diffeomorphically onto $c \setminus \Sigma$. In [13] we prove that c^1 is analytic.

It is worth pointing out that if $c = c(t)$ is parameterized by the parameter t, then at regular points t (where $dc/dt(t) \neq 0$) we have $c^1(t) = (c(t), \text{span}\ \{dc(t)/dt\})$. At singular points t_* (where $dc/dt(t_*) = 0$) we have $c^1(t_*) = \lim_{t \to t_*} c^1(t)$.

If c is tangent to a rank 2 (complex) distribution $D \subset TM$ then its prolongation c^1 must lie in the projectivization $\mathbb{P}D \subset \mathbb{P}TM$ of D. The space $\mathbb{P}D$ is a bundle over

M with fiber the complex projective line. Now $\mathbb{P}D$, viewed as a complex manifold, is itself endowed with a canonical rank 2 (complex) distribution which we denote D^1, and call the prolongation of D. We may define D^1 by

$$D^1(p) = (d\pi_m)^{-1}(\ell), p = (m, \ell) \in \mathbb{P}D.$$

Here $\pi : \mathbb{P}D \to M$ is the projection sending (m, ℓ) to s. Alternatively, a smooth curve γ in $\mathbb{P}D$ consists of a moving point and a moving line $\gamma(t) = (m(t), \ell(t))$ and we can define D^1 by declaring that

$$\text{a curve } \gamma(t) = (m(t), \ell(t)) \text{ is tangent to } D^1 \text{ iff } dm(t)/dt \in \ell(t).$$

Now c^1 is tangent to D^1 at every point over $c \setminus \Sigma$. It is in fact tangent to D^1 at all of its points, by continuity of D and the analyticity of c. Thus, we can repeat the procedure, to achieve the second prolongation $c^2 = (c^1)^1$ tangent to a rank 2 distribution $D_2 = (D^1)^1$ on the manifold $M_2 = \mathbb{P}(D^1)$. Continuing in this manner we get a sequence of prolonged curves c^k tangent to rank 2 distributions D_k on manifolds M_k. The M_k , $k = 1, 2, \ldots$ form a tower of $\mathbb{CP}^1$-bundles

$$\ldots \to M_{k+1} \to M_k \to M_{k-1} \to \ldots M_1 = \mathbb{P}(D) \to M.$$

We apply this construction to a planar curve $c \subset \mathbb{C}^2$, assumed analytic. The curve is trivially tangent to the tangent bundle $\Delta_0 := T\mathbb{C}^2$ of $\mathbb{C}^2$, a rank 2 distribution. The first prolongation of the triple $(c, \Delta_0, \mathbb{C}^2)$ consists of an integral curve c^1, a rank 2 distribution $\Delta_1 = (\Delta_0)^1$, and a (complex) 3-dimensional manifold $\mathbb{P}T\mathbb{C}^2$ which supports both Δ_1 and c. The points of $\mathbb{P}T\mathbb{C}^2$ are the marked lines described in the introduction. Δ_1 is a contact distribution. Iterating the prolongation construction we obtain $(c^j, \Delta_j, P^j(\mathbb{C}^2))$. The $P^j(\mathbb{C}^2)$ fit together to form a tower of $\mathbb{P}^1$-bundles

$$\ldots \to P^{j+1}(\mathbb{C}^2) \to P^j(\mathbb{C}^2) \to \ldots \to P^1(\mathbb{C}^2) = \mathbb{P}T\mathbb{C}^2 \to \mathbb{C}^2.$$

Each $P^j(\mathbb{C}^2)$ is endowed with its rank 2 distribution Δ_j, and $P^{j+1}(\mathbb{C}^2)$ is the total space of the projectivized bundle $\mathbb{P}(\Delta_j)$. Note that a point $p_{j+1} \in P^{j+1}(\mathbb{C}^2)$ is to be viewed as a pair (p_j, ℓ) with $p_j \in P^j(\mathbb{C}^2)$ and $\ell \subset \Delta_j(p_j)$ a line.

We call this tower of $\mathbb{P}^1$ *bundles, endowed with their distributions "the Monster tower", see* [12]*,* [13]*. When we say we are at level* j *we mean we are working within* $(P^j(\mathbb{C}^2), \Delta_j)$*.*

The curves $c^j \subset P^j(\mathbb{C}^2)$ are tangent to Δ_j and are defined iteratively by $c^{j+1} = (c^j)^1$. They are all analytic. (See [13]).

The real version of $P^1(\mathbb{C}^2)$ occurs frequently in books and papers of Arnol'd. The real version $P^2(\mathbb{C}^2)$ occurs infrequently and is the primary example of an "Engel manifold". One place it occurs in Arnol'd's works is [1], in section 9.

Example 3.1 (The cubic cusp)**.** *The curve* $x^3 = y^2$ is parameterized as $x = t^2, y = t^3$. Introduce the fiber coordinate u on $P^1(\mathbb{C}^2)$ by setting $[dx, dy] = [1, u]$ which is to say $u = dy/dx$ is the slope of the tangent curve. For the cubic cusp $u = 3t^2dt/2tdt = (3/2)t$. The prolonged curve c^1 is then given in coordinates (x, y, u) by $(t^2, t^3, (3/2)t)$ and is a smooth immersed curve.

Example 3.2 (The A_4 singularity)**.** The curve $x^5 = y^2$ is parameterized as $c(t) = (t^2, t^5)$. We compute, as in the previous example, that in the coordinates $x, y, u_1 = dy/dx$ its first prolongation c^1 is $(t^2, t^5, (5/2)t^3)$ which is singular. Its second prolongation c^2 is immersed. To see this introduce the fiber coordinate u_2 on $P^2(\mathbb{C}^2)$, near the point $c^2(0) = (0, 0, 0, \text{span }\{\partial/\partial x\})$ by setting $[dx, du_1] = [1, u_2]$ which is to say $u_2 = du_1/dx$. In the coordinates (x, y, u_1, u_2) the second prolongation c^2 is given by $(t^2, t^5, (5/2)t^3, (15/4)t)$.

Example 3.3 (The A_{2k} singularity)**.** Consider the curve $c(t) = (t^2, t^{2k+1})$. Following the same lines we find that it takes k prolongations to get an immersed curve; that is, $c^k(t)$ is immersed at $t = 0$ while the $c^j(t)$'s, $j < k$, are not.

Example 3.4 (The E_6 singularity)**.** Let $c(t) = (t^3, t^4)$. Introduce the fiber coordinate $u = dy/dx$ as in Example 3.1. The prolonged curve c^1 is then given in coordinates (x, y, u) by $(t^3, t^4, (4/3)t)$ and is a smooth immersed curve.

Example 3.5 (The E_8 singularity)**.** Let $c(t) = (t^3, t^5)$. In the coordinates (x, y, u_1), $u_1 = dy/dx$, the curve c^1 is given by $(t^3, t^5, (5/3)t^2)$ which is singular. Introduce the coordinate u_2 on $P^2(\mathbb{C}^2)$, near the point $c^2(0) = (0, 0, 0, \text{span }\{\partial/\partial u_1\})$ by setting $[du_1, dx] = [1, u_2]$ which is to say $u_2 = dx/du_1$. Note that this u_2 is *not the same* u_2 *coordinate as in Example 3.2.* In these coordinates (x, y, u_1, u_2) the second prolongation c^2 is given by $(t^3, t^5, (5/3)t^2, (9/10)t)$ and is a smooth immersed curve.

These examples illustrate:

Proposition 3.6 (Nobile [14], see also [13])**.** *Let c be an analytic plane curve germ. Then there is a finite number j such that the jth prolongation $c^j \subset P^j(\mathbb{C}^2)$ is a nonsingular curve germ.*

As discussed in section 1, Nobile proved this proposition in [14]. It appears there as "Corollary 1". Two of us reproved this proposition for unibranched planar singularities in [13]. As we've already discussed, our treatment contains many things not found in Nobile: the contact/Legendrian nature of the prolongation process, a formula for the minimum number j to resolution in terms of the Puiseux characteristic of the curve, and a way to understand the details of how the prolonged curve touch the critical curves defined in the next section. These details are necessary to build the prolongation resolution graph of the singularity, and hence to even formulate our main theorem here.

4. Critical curves

Proposition 3.6 guarantees that the jth prolongation c^j of an analytic plane curve germ c is non-singular for large enough j. If we stopped at the first such j and simply compared this prolonged curve $c^j(t)$ with the resolution of c by blow-up (its proper transform) our story would be uninteresting. We would have two immersed curves, albeit in spaces of different dimensions. Any two immersed curve germs are equivalent from the viewpoint of local analytic geometry: they look like one of the coordinate axes in some coordinate system. We would not have anything interesting to compare beyond how many steps are required to resolution in the two cases.

What makes blow-up of a planar curve germ c interesting is the exceptional curves. These are projective lines added to the curve with each blow-up. Together with the proper transform of c they form a "multi-curve" in a surface : a finite union of curves in a surface whose union is the resolved curve of blow-up. The intersections among the components of this multi-curve define a graph intrinsically related to the curve germ. *It is this graph that we want to "see" in prolongation.*

In order to see it we need prolongation analogues of the exceptional fibers. These analogues are called "critical curves".

Definition 4.1. A critical curve in $P^j(\mathbb{C}^2)$, $j > 0$ is an embedded integral curve for Δ_j whose projection to the plane $\mathbb{C}^2$ is a constant curve.

The simplest critical curves are the vertical curves.

Definition 4.2. The vertical curves at level j are the fibers of the projection $P^j(\mathbb{C}^2) \to P^{j-1}(\mathbb{C}^2)$. Such a curve will be denoted V_j.

Remark 4.3 (Warning)**.** The definitions of critical curve and of vertical curves which we have just given differ at level 1 from the definitions in [13]. In [13] we do not consider the vertical curves V_1 at level 1, or its prolongations, to be critical curves. All the other $V_j, j > 1$ and their prolongations comprise the critical curves of [13]. See section 10 for more on this difference.

We can view a vertical curve as the prolongation of the point over which it lies. For example, think of the origin in $\mathbb{C}^2$ as the image of the constant curve $t \mapsto 0$. Every line through 0 is tangent to this curve, so the prolongation of 0, viewed as a constant curve, is the vertical curve over 0, i.e. the fiber over 0 for the fibration $P(\mathbb{C}^2) \to \mathbb{C}^2$. It is a copy of $\mathbb{CP}^1$ in $P(\mathbb{C}^2)$.

Definition 4.4. A tangency curve is the prolongation $(V_i)^j$, $j > 0$ of some vertical curve V_i, $i \geq 1$.

For the rationale between the terminology "tangency" see [13].

Proposition 4.5. *Let γ be a critical curve. Then γ is either a vertical curve, or a tangency curve. Tangency curves are not vertical curves.*

Proof. The proof of the corresponding proposition in [13] holds here. (See the Remark 4.3, and section 10 concerning the difference between critical curves there and here). □

5. Resolution by prolongations

Start with our singular curve $c \subset \mathbb{C}^2$. Add to the prolongation c^1 of c the prolongation of each of c's singular points of c, in this way adding a finite collection of vertical curves to the old prolongation c^1. The resulting collection of curves is called the full (first) prolongation

Iterate this construction, forming $P(c), P^2(c), \ldots, P^j(c) \subset P^j(\mathbb{C}^2)$. The different branches of $P^j(c)$ consist of the old prolongation c^j and a finite collection of critical curves.

Consider the case of a unibranched curve germ $c = c(t)$. Suppose that its $j-1$st prolongation c^{j-1} is singular, with singular point $p_{j-1} = c^{j-1}(0)$. When we prolong again to form c^j we must add the vertical curve $V_j = p_j^1$ at level j. We carry along with us the previously introduced critical curves, by prolonging them. Since we add exactly one new critical curve upon each prolongation, $P^j(c)$ consists of $j+1$ curves, these being c^j and the j critical curves, $V_j, V_{j-1}^1, V_{j-1}^2, \ldots, V_1^{j-1}$.

At what step j do we declare the multi-curve $P^j(c)$ to be "resolved"?

Definition 5.1. A finite collection of embedded integral curves for a rank 2 distribution D is said to form a 'normal system" if, whenever two curves intersect at a point p , their tangent lines intersect *transversally* within $D(p)$, and no three curves intersect at a single point.

If c^j is tangent to V_j, or to a V^i_{j-i} then we will count that tangency point as a critical point even if c^j is immersed. Similarly if c^j is immersed but forms a triple point with two of the critical curves, we count that triple point as a singular point and we continue the prolongation process as before. (If two critical curves intersect, then their intersection is transverse within Δ_j, so that tangencies between critical curves cannot occur.)

Definition 5.2. We will say that the unibranched singularity c has been resolved by prolongation when c^j is immersed and $P^j(c) = c^j \cup V_j \cup V^1_{j-1} \cup \ldots \cup V^{j-1}_1$ forms a normal system of curves for Δ_j.

Example 5.3 (Resolution by prolongation of a cusp)**.** We return to the cubic cusp $c(t) = (x(t), y(t)) = (t^2, t^3)$. We saw (Example 3.1) that its first prolongation is coordinatized as $(t^2, t^3, \frac{3}{2}t)$ with the last coordinate representing $u_1 = dy/dx$. In these coordinates the vertical curve V_1 is $(0, 0, t)$. We see that c^1, though immersed, is tangent to the vertical curve. So we must prolong again. This is done by introducing the new coordinate $u_2 = dx/du_1$ which represents a fiber affine coordinate on $P^2(\mathbb{C}^2) \to P^1(\mathbb{C}^2)$. In the x, y, u_1, u_2 coordinates we find that $c^2 = (t^2, t^3, \frac{3}{2}t, \frac{4}{3}t)$ while $V^1_1 = (0, 0, t, 0)$, and V_2, the new vertical curve is given by $(0, 0, 0, t)$. The distribution Δ_2 at level 2 is given in these coordinates by $dy - u_1 dx = 0$ and $dx - u_2 du_1 = 0$. At $t = 0$ all three curves pass through the coordinate origin, and their tangents form three distinct lines, $du_1 = 0, du_2 = 0$ and $du_1 = du_2$ within $\Delta_2(0, 0, 0, 0)$. We have a triple intersection. One more prolongation is required to resolve the singularity according to the definition. We find that $P^3(c) = c^3 \cup V^3_1 \cup V^1_2 \cup V_3$. At level 3, we have that c^3 and V_3 intersect transversally, and V^3_1, V^1_2 intersect none of the other curves.

Theorem 5.4. *Any well-parameterized curve germ can be resolved by prolongation in a finite number r of steps.*

Proof. The theorem is almost completely proved in [13]. We prove there that for any such curve germ c, there is a finite number k (depending only on c's Puiseux characteristic) such that after k prolongations c^k becomes immersed and regular, where "regular" means that c^k is not tangent to a critical curve. For $j < k$ the c^k are either not immersed, or are tangent to a critical direction. At step k we are in a situation identical to the penultimate step in the cubic cusp example just presented: c^k forms a triple intersection with the vertical curve and a tangency curve. One more prolongation yields the resolution in the sense of Definition 5.2. Thus the r of Theorem 5.4 is $k+1$ where k is the regularization number of [13]. □

6. Main theorem

Definition 5.2 was made in analogy with the definitions in blow-up where the jth blow-up $B^j(c)$ of c consists of the proper transform of c, and j exceptional curves $E_1, \ldots, E_j$, each one an embedded $\mathbb{CP}^1$. These curves all lie on a surface X_j. We declare the curve to be resolved by blow-up when the component curves of $B^j(c)$

form a normal system for the tangent bundle $D = T(X_j)$, in the sense of Definition 5.2.

There is a standard way to draw a graph, sometimes called the "dual graph", which encodes the combinatorial relationships among component curves in resolution by blow-up. (See for example [8]. The dual graph is dual to the 'diagram' there.) The vertices of the dual graph are the exceptional curves. Two vertices are connected by an edge if and only if they intersect. Finally, there is an arrow representing the proper transform and this arrow connects to the exceptional curve to which it intersects, this curve always being the last occurring exceptional curve.

If we associate a graph to the normal system in resolution by prolongation in this same way, we will obtain at a dual graph having little relation to the graph for blow-up. What one finds is that most of the critical curves intersect no other critical curves within the prolongation, and hence most of the critical curves correspond to isolated vertices. On the other hand, in blow-up, every exceptional curve intersects some other exceptional curve, and one finds that the blow-up graph is connected. The discrepancy between the two graphs is a direct consequence of the difference between what happens to a pair of transversally intersecting curves when we prolong, versus when we blow up. When we prolong two integral curves which intersect transversally within Δ_j, the resulting curves do not intersect at all. On the other hand, when we blow-up two curves which intersect transversally at *any point besides the point which is the center of the blow-up operation*, then the resulting curves continue to intersect transversally.

To get the correct diagram for prolongation we must alter our definition of what it means for two component curves to "intersect". We call the new relation of intersection "incidence". To present the definition of incidence, we first introduce a notational labelling conventions for the critical curves of $P^j(c)$.

Notational convention. Let V_r^{j-r} be one of the critical curves comprising $P^j(c)$. We will use the symbol V_r to denote this curve, viewed at any level of the monster. Since the curve first arises at level r, it is declared to be the empty curve when viewed at levels $k < r$, that is , we declare, for incidence counting purposes, that V_r, viewed in $P^k(\mathbb{C}^2), k < r$ is the empty curve. When we view V_r at level $k > r$ we mean the $k - r$-fold prolongation V_r^{k-r} of V_r. Similarly, when we say we are viewing c^j at level $i < j$ we are speaking of the ith prolongation c^i.

Using this notation, we have that $P^j(c) = c^j \cup V_j \cup V_{j-1} \cup \ldots \cup V_1$.

Definition 6.1. We declare that two component curves A, B of $P^j(c)$ to be incident if, for some $i \leq j$ they intersect normally within $P^i(c)$. In other words, when viewed at level $i \leq j$ the curves A and B intersect transversally at some point $q \in P^i(\mathbb{C}^2)$, and no other component of $P^i(c)$ passes through q.

Theorem 6.2 (Main Theorem)**.** *Let c be a unibranched plane curve singularity. Use Definitions 5.1 and 6.1 for resolution by prolongation. Then the graph for resolution of c by prolongation is isomorphic to the graph for resolution by blow-up.*

The rest of the paper is devoted to proving this theorem. We need to develop more machinery.

7. Regular, vertical, and tangency directions and points

Definition 7.1 (of Regular, Vertical, Tangency)**.**

• We will call a direction $\ell \subset \Delta_j(m)$, $j \geq 1$ the vertical direction if it is tangent to the vertical curve through m. See Definition 4.2.
• We call the direction a tangency direction if it it is tangent to a tangency curve through m. See Definition 4.4.
• We call the direction a critical direction if it is tangent to a critical curve passing through m, that is, if it is either a vertical or a tangency direction. See Prop. 4.5.
• If the line is not critical, then we call ℓ a regular direction.
• We call $p = (m, \ell) \in P^{j+1}(\mathbb{C}^2)$ a vertical point if the line $\ell \subset \Delta_j(m)$ is a vertical direction.
• We call the point a tangency point if the line is a tangency direction.
• We call the point a critical point if the line is a critical direction.
• We call the point a regular point if ℓ is a regular direction.

Proposition 7.2. *Let $p \in P^j(\mathbb{C}^2)$. If p is a regular point then there is exactly one critical direction in the plane $\Delta_j(p)$: the vertical direction. If p is a critical point then there are exactly two critical directions in the plane $\Delta_j(p)$: the vertical line and the tangency line, and these two lines intersect transversally within $\Delta_j(p)$.*

Proof. The proof of the analogous Proposition from [13], Proposition 2.41 there, holds in our situation. □

8. Incident relations

As a consequence of Proposition 7.2 we have:

Proposition 8.1. *Critical curves obey the following intersection rules:*

• (a) *at most two critical curves can intersect at a point.*

• (b) *if two critical curves intersect at a point, their intersection is transverse, and one of the two curves is vertical.*

• (c) *The vertical curve V_i through p_i at level i, $i > 1$ always intersects V_{i-1} (see the Notational Convention of section 6). If $p_{i-1} = \pi_{i,i-1}(p_i)$ is regular then V_i intersects no other critical curve. If p_{i-1} is critical then V_i intersects precisely one other critical curve, namely the tangency curve through p_{i-1}.*

Proof. The first two facts of Proposition 8.1 follow immediately from the preceding proposition, Proposition 7.2. To see the validity of the last fact (c), project the system of critical curves intersecting V_i one level down, to get critical curves passing through $p = p_{i-1}$. They are tangent to critical directions. The statement now follows directly from Proposition 7.2. □

Suppose that V_i intersects some $V_j, j < i$ at level i. Are V_i and V_j incident in the diagram of resolution? According to the above facts, if c^i is deleted from the system of curves $P^i(c)$, the intersection of V_i and V_j is normal. Only c^i, by passing through $q = V_i \cap V_j$ can destroy this incidence. That is to say: V_i is incident to V_j if and only if $c^i(0) \neq V_i \cap V_j$. At any level i we have that V_i and c^i intersect at $c^i(0)$. It follows that at the resolution level j we have c^j incident to V_j alone.

We can summarize the incidence relations as follows.

Proposition 8.2. *Let j be the resolution level for the unibranched planar curve germ $c(t)$ and $P^j(c) = c^j \cup V_j \cup V_{j-1} \cup \ldots \cup V_1$ the corresponding resolution by prolongation. Then the incidence relations among the components of $P^j(c)$ are as follows. Set $m(i,k) = max\{i,k\}$. V_k is incident to V_i if and only if V_k and V_i intersect at level $m = m(i,k)$ and this intersection point is not $c^m(0)$. And V_i intersects c^j if and only if $i = j$.*

9. RVT codes

Definition 9.1 (RVT code)**.** For $p \in P^i(\mathbb{C}^2)$, $i > 1$ write p_j for the projection of p to lower levels j, $j = 0, 1, 2, \ldots, i-1$, setting $p_i = p$ for consistency. Assign to each such level j with $2 \leq j \leq i$ the letter R, V, or T depending on whether or not p_j is a regular (R) , vertical (V) or tangency point (T). Write $\omega_j = \omega_j(p)$ for this letter. Then the RVT code of p is the word $\omega = \omega(p) = \omega_2\omega_3 \ldots \omega_k$, comprised of these letters in order of appearance.

According to Proposition 7.2, the letter T cannot immediately follow an R. This is the only restriction we need to impose on a word consisting of the letters R, V, and T in order for it to be the RVT code of some point of the Monster. We call this restriction the "grammar rule". We formalize the notion:

Definition 9.2. An RVT code is a finite sequence of letters from the collection R, V, T beginning with an R or V, and subject to the further constraint that whenever a T appears, it must be preceded by either a T or a V. Such a code is written $\omega = \omega_2\omega_3 \ldots \omega_k$, so that $\omega_i \in \{\mathrm{R}, \mathrm{V}, \mathrm{T}\}$.

We now assign RVT codes to analytic plane curve germ, following the lines of [13], particularly, section 3.7 there.

Theorem 9.3. *Let c be a well-parameterized singular analytic curve germ. Then there exists a k such that $c^k(t)$ is immersed and tangent to a regular direction. For all $j > k$ we have $\omega_j(c^j(0)) = \mathrm{R}$. If k is the smallest such integer, then $c^k(0)$ is a critical point.*

Proof. The proof of the analogous theorem from [13], Theorem 2.36, holds in our setting. □

Definition 9.4. The regularization level of the planar curve germ $c(t)$ is the smallest integer k such that $c^k(t)$ is immersed and tangent to a regular direction at $t = 0$.

Definition 9.5. The RVT code of c is the RVT code $\mathrm{R}\omega_2 \ldots \omega_\mathrm{k}$ where $\omega_2 \ldots \omega_k$ is the $c^k(0)$ and k is the regularization level of c.

Rationale for the initial R . There are no special directions in the plane, so all points at level 1 are viewed as regular (R) points.

Theorem 9.6. *The number of prolongations needed to resolve c is $k+1$ where k is its regularization level.*

Proof. This is essentially what we proved in proving Theorem 5.4 above. Since $c^k(0)$ is critical (see Theorem 9.3) there are, according to Proposition 7.2, two critical curves pass through it, the vertical curve, and a tangency curve, and these form a triple point at level k , with three distinct tangents. □

Notation. If $\omega_i(c)$ is a critical letter, meaning either V or T, then by Proposition 7.2 precisely two critical curves pass through $c^i(0)$. One of these is V_i. The other is of the form V_a for some unique $a = a(i) < i-1$, which is a tangency curve. The integer $a(i)$ depends only on $c^i(0)$.

Theorem 9.7. *The* RVT *code of* $\mathrm{R}\omega_2\omega_3\ldots\omega_\mathrm{k}$ *of a plane analytic curve germ* c *determines the incidence relation of its resolution by prolongation, according to the following rules. Add a final* $\omega_{k+1} = \mathrm{R}$ *to the end of the word. Let* C *below stand for either critical letter* Vor T. Use the notation $a(i-1)$ from above when $\omega_{i-1} = \mathrm{C}$, and $i \geq 2$. Then

- if $\omega_{i-1}\omega_i = \mathrm{RV}$ then V_i is incident to no curve V_j with $j < i$.
- if $\omega_{i-1}\omega_i = \mathrm{C}V$ then V_i is incident to precisely one curve $V_j, j < i$, namely the curve corresponding to $j = a(i-1)$.
- if $\omega_{i-1}\omega_i = \mathrm{CT}$ then V_i is incident to V_{i-1} and no other curve $V_j, j < i$.
- If $\omega_{i-1}\omega_i = \mathrm{CR}$ then V_i is incident to precisely two curves V_j with $j < i$, namely the curves correspond to $j = i-1$ and $j = a(i-1)$
- If $\omega_{i-1}\omega_i = \mathrm{RR}$ then V_i is incident V_{i-1} and to no other $V_j, j < i$.

Proof. The proof is based on Propositions 8.1 and 8.2. We proceed by cases according to the value of the letter ω_i.

If $\omega_i = \mathrm{V}$, then $c^{i-1}(t)$ is tangent to the vertical curve V_{i-1} at level $i-1$. Upon prolongation this means that $c^i(t)$ intersects V_{i-1} in addition to V_i, and so we do not count V_i as being incident to V_{i-1}. If, in addition to $\omega_i = \mathrm{V}$ we have that $\omega_{i-1} = \mathrm{R}$, then, according to (c) of Proposition 8.1, the only critical curve intersecting V_i is V_{i-1}, so that V_i is not incident to any V_j with $j < i$. If instead ω_{i-1} is critical then V_i intersects two critical curves, V_{i-1} and $V_{a(i-1)}$. Only one of these, V_{i-1} also intersects c^i, since the two critical directions at level $i-1$ are distinct. So V_i intersects $V_{a(i-1)}$ in this case.

If $\omega_i = \mathrm{T}$ then $c_{i-1}(t)$ must be tangent to the tangency curve $V_{a(i-1)}$ at level $i-1$, and so $V_{a(i-1)}$, c^i and V_i all intersect at level i. Thus we do not count V_i as being incident to the tangency curve $V_{a(i-1)}$. But the vertical curve at level $i-1$ is not tangent to $c^{i-1}(t)$, so that $c^i(t)$ does not intersect V_{i-1}. Thus we do count V_i as being incident to V_{i-1}. This is the only V_j, $j < i$ that V_i can be incident to . (Recall that when $\omega_i = \mathrm{T}$ we must ω_{i-1} critical.)

If $\omega_i = \mathrm{R}$ then $c^{i-1}(t)$ is not tangent to any critical direction, so that $c^i(t)$ does not intersect any $V_j, j < i$ at level i. If $\omega_{i-1} = \mathrm{R}$ then there is exactly one curve V_j, $j < i$ that intersects V_i at level i, namely V_{i-1} (and their intersection point is not $c^i(0)$). Otherwise, ω_{i-1} is critical and there are exactly two critical curves V_j, $j < i$ whose prolongations intersect V_i, namely V_{i-1} and $V_{a(i-1)}$. □

10. Contact versus surface symmetries and RVT codes

The RVT codes introduced in the previous section differ slightly from the RVT codes defined in [13]. We need this variation because vertical curves at level 1 are considered to be critical curves here, but not in [13]. Correspondingly, the RVT codes here are indexed starting at level 2, while the codes of [13] are indexed starting at level 3.

Symmetry considerations best explain these variations in what it means to be a critical curve, and the consequent variations in coding. The relevant symmetry group of [13] is the full symmetry group of the distribution Δ_j, which was proved to be the group of contact transformations at level 1, lifted to level j. On the other hand, the relevent symmetry group for the purpose of the present paper is the group of planar diffeomorphisms, lifted to act on the jth level of the Monster. This planar group is much smaller than the group of contact transformations. Indeed most contact transformations at level 1 do not map vertical curves to vertical curves, while the planar diffeomorphism group sits inside the contact transformation group at level 1 precisely as the subgroup which preserves the set of vertical curves. To re-iterate, from the perspective of [13] vertical curves at level 1 have no invariant meaning and so are excluded from the club of critical curves, while in the context of the current paper, vertical curves at level 1 have invariant meaning.

In [13] we constructed a bijection between the RVT codes and those Puiseux characteristics $[\lambda_0, \lambda_1, \ldots, \lambda_m]$ satisfying $\lambda_1 > 2\lambda_0$. The constraint $\lambda_1 > 2\lambda_0$ is related to excluding the vertical directions at level 1 from being critical. (The prolongation of a curve with Puiseux characteristic having $\lambda_1 < 2\lambda_0$ is tangent to the vertical direction at level 1.) As a welcome and necessary consequence of including vertical directions at level 1 among critical curves, we have no constraints on the Puiseux characteristics in the present paper, and the bijection now holds between our variant RVT codes and the space of all Puiseux characteristics .

11. Computing RVT codes

There are two ways to compute the RVT code of a plane curve germ. One way uses the Puiseux characteristic of the curve. The other way uses a special atlas of coordinates on the Monster which are detailed in [13]. We will not be using the first way, but mention it now in passing. For the definition of the Puiseux characteristic and its properties, we refer the reader to [16].

Theorem 11.1. *Two well-parameterized plane curve germs having the same Puiseux characteristic have the same* RVT *code. Two well-parameterized plane curve germs having the same* RVT *code have the same Puiseux characteristic .*

This theorem implies the existence of a bijection between RVT codes and Puiseux characteristics. Such a bijection is constructed by recursion and can be found in [13]. A slight variation in that recursion formula is needed due to the difference between our RVT codes and those of [13]. See section 10 above.

We proceed with the second method of calculation, refined by way of an operator we call directional blow-up and introduced (again with a slight variation) in [13]. Let $\mathcal{C}$ denote the set of analytic plane curve germs passing through the origin and not tangent to the y-axis:

$$\mathcal{C} = \big\{(x(t), y(t) : \quad x(0) = y(0) = 0, \ \operatorname{ord} x(t) \leq \operatorname{ord} y(t)\big\}.$$

Here $\operatorname{ord} f(t) = r$ if $f(t) = at^r + \text{h.o.t.}$, $a \neq 0$. For $(x(t), y(t)) \in \mathcal{C}$ set

$$z(t) = z[x, y](t) = \frac{y'(t)}{x'(t)} - \Big(\frac{y'(t)}{x'(t)}\Big)(0)$$

so as to define an operator

$$\mathcal{P} : \mathcal{C} \to \mathcal{C}$$

by

$$\mathcal{P}((x(t),y(t))) = \begin{cases} (x(t),z(t)) \text{ if } \operatorname{ord} x(t) \le \operatorname{ord} z(t) \\ (z(t),x(t)) \text{ if } \operatorname{ord} x(t) > \operatorname{ord} z(t) \end{cases}$$

The operator $\mathcal{P}$ can be iterated, allowing us to define a sequence of curve germs and function germs:

$$\mathcal{P}^i(c) = ((x_i(t),y_i(t)), \qquad z_i(t) = \frac{y_i'(t)}{x_i'(t)} - \Big(\frac{y_i'(t)}{x_i'(t)}\Big)(0), \quad i = 1,2,\ldots.$$

The following two proposition are proved (with a slight alteration as per section 10) in [13].

Proposition 11.2. *Let* $(x_0(t),y_0(t)) \in \mathcal{C}$ *be a well-parameterized curve germ with* RVT *code* $\mathrm{R}\ell_1 \ldots \ell_{\mathrm{k}-1}$ *and let* $(x_i,y_i) = \mathcal{P}^i(x_0,y_0)$*, and* z_i *be as above. Then*

$$\ell_{i+1} = \mathrm{R} \implies \ell_{\mathrm{i}+2} = \begin{cases} \mathrm{R} \text{ if } \operatorname{ord} x_i(t) \le \operatorname{ord} z_i(t) \\ \mathrm{V} \text{ if } \operatorname{ord} x_i(t) > \operatorname{ord} z_i(t) \end{cases}$$

$$\ell_{i+1} = \mathrm{V} \textit{ or } \mathrm{T} \implies \ell_{i+2} = \begin{cases} \mathrm{T} \text{ if } \operatorname{ord}\ x_i(t) < \operatorname{ord}\ z_i(t) \\ \mathrm{R} \text{ if } \operatorname{ord} x_i(t) = \operatorname{ord} z_i(t) \\ \mathrm{V} \text{ if } \operatorname{ord} x_i(t) > \operatorname{ord} z_i(t). \end{cases}$$

We can also use directional blow-up to see when the resolution-by-prolongation procedure halts:

Proposition 11.3. *The curve regularizes at level* $k+1$ *where* k *is the first integer such that* $\operatorname{ord}(x_k) = \operatorname{ord}(y_k) = 1$.

Example 11.4. Some examples of RVT codes obtained using Proposition 11.2 are given in the table below.

TABLE 1. Examples of RVT codes

Singularity	Represented by the curve	RVT code
A_{2k}	(t^2, t^{2k+1})	$\mathrm{R^kV}$
E_6	(t^3, t^4)	VT
E_8	(t^3, t^5)	VV
$[6; 14, 15]$	$(t^6, t^{14} + t^{15})$	RRVTRV

Let us use Propositions 11.2 and 11.3 to check the last row of the table, for the curve $(x_0,y_0) = (t^6, t^{14}+t^{15})$ which represents the Puiseux characteristic $[6,14,15]$. We compute

$$z_0 = k_1t^8 + k_2t^9 \implies (x_1,y_1) = (t^6,\ k_1t^8 + k_2t^9).$$

Here, and in the following $k_1,k_2,\ldots,$ a,b,c,d denote non-zero constants. Thus $\ell_2 = \mathrm{R}$. At the next iteration

$$z_1 = k_3t^2 + k_4t^3,\ \operatorname{ord}(z_1) = 2 < \operatorname{ord}(x_1) = 6 \implies$$
$$\ell_3 = \mathrm{V}, \quad (x_2,y_2) = (k_3t^2 + k_4t^3,\ t^6).$$

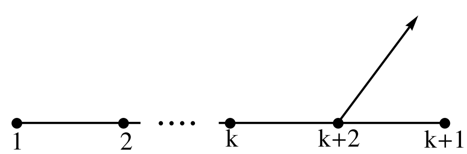
At the next iteration we have

$$z_2 = 6t^5/(2k_3t + 3k_4t^2) = ct^4 + dt^5 + \cdots \implies$$
$$\ell_4 = \mathrm{T}, \quad (x_3, y_3) = (k_3t^2 + k_4t^3,\ ct^4 + dt^5 + ...).$$

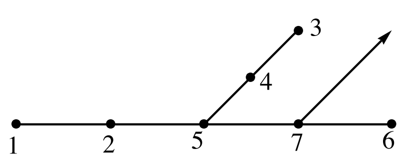
Now $z_3 = (4ct^3 + 5dt^4 + \ldots)/(2k_3t + 3k_4t^2) = k_5t^2 + k_6t^3 + \ldots$. We have $\mathrm{ord}(z_3) = \mathrm{ord}(x_3)$ so that according to Proposition 11.2 we have that $\ell_5 = \mathrm{R}$. Now $(x_4, y_4) = (k_3t^2 + k_4t^3, k_5t^2 + k_6t^3 + \ldots)$. At the next step it is important that there is no cancellation, so that in forming $z_4 = y_4'/x_4' - (y_4'/x_4')(0)$ one has $z_4'(0) \neq 0$. To check this 'no cancellation' requires keeping some track of coefficients, enough to verify that (k_3, k_4) and (k_5, k_6) are linearly independent. They are. Then $z_4 = at + bt^2 + \ldots$. We have $\mathrm{ord}(z_4) = 1 < 2 = \mathrm{ord}(x_4)$ and so $\ell_6 = \mathrm{V}$ and $(x_5, y_5) = (at + bt^2 + \ldots, k_3t^2 + k_4t^3)$. Iterating again, $(x_6, y_6) = (k_7t + \ldots, k_8t)$ (Again one must check, via coefficients, that there is no cancellation.) By Proposition 11.3 we are done. Collecting all the ℓ's we see that the RVT code of this curve is RRVTRV.

From the RVT code and Proposition 9.7 we can read off the incidence diagram of the prolongation resolution of the curve.

Example 11.5. The fact that the RVT code of the A_{2k} is $\mathrm{R}^k\mathrm{V}$, the RVT code of the singularity $[6; 14, 15]$ is RRVTRV and Proposition 9.7 imply that the incidence diagram of the prolongation resolution of these singularities are given by the following graphs:



The incidence diagram of the prolongation resolution of A_{2k}



The incidence diagram of the prolongation resolution of a curve with the Puiseux characteristic $[6; 14, 15]$.

12. Blow-up

We review the method of blow-up to resolve plane curve singularities. We set up notation to facilitate the proof of the main theorem. There are numerous good references for this section, including [2], [7], [8], [16]. The notion of the blow-up

RVT code we give here is apparently new, but appears to be closely linked to the notion of proximity among infinitely near points.

Constructing blow-ups. The blow-up of the plane at the origin can be realized as that subvariety $Bl_0(\mathbb{C}^2)$ of $P^1(\mathbb{C}^2)$ consisting of those pairs $(p, \ell) \in P^1(\mathbb{C}^2)$ for which ℓ passes through the origin 0 as well as through p. The restriction of the natural projection $P^1(\mathbb{C}^2) \to \mathbb{C}^2$ to $Bl_0\mathbb{C}^2$ is called the "blow-down map" and denoted $\beta : Bl_0\mathbb{C}^2 \to \mathbb{C}^2$: $\beta(p, \ell) = p$. We set $E = \beta^{-1}(0)$ and call E the exceptional curve. E is an embedded copy of $\mathbb{CP}^1$. It coincides with the vertical curve V_1 over 0. Away from E the blow-down map is a diffeomorphism since if $\beta(p, \ell) = p$, and $p \neq 0$ then $\ell = span(p)$.

Coordinates on the Blow-up. Away from the exceptional fiber E planar coordinates (x, y) coordinatize the blow-up since β is a diffeomorphism off E. To cover points of E we need two coordinate charts. Take $p_1 = ((0,0), \ell_0) \in E$ and suppose $\ell_0 \neq y-$axis. As a neighborhood of p_1 consider all points $((x, y), \ell)$ of the blow-up for which $\ell \neq y-$axis. Use affine coordinate w_1 for these lines: $\ell = [1, w_1]$ and $p_1 = ((0,0), [1, w_1])$. Then (x, w_1) coordinatize our neighborhood . The condition defining $Bl_0(\mathbb{C}^2)$ is that $[x, y] = [1, w_1]$ for $(x, y) \neq (0,0)$, which is to say that $y = xw_1$ and so the blow-down map is

$$\beta : (x, w_1) \mapsto (x, xw_1) = (x, y).$$

In the coordinates (x, w_1) the exceptional curve E is defined by $x = 0$.

Notation. By a slight abuse of notation we will say that w_1 is defined by the equation

$$w_1 = y/x \tag{12.1}$$

which is valid for $x \neq 0$. Even though equation (12.1) does not make sense on the exceptional curve E, there is no ambiguity in our original definition of w_1 and that equation uniquely picks out w_1 as an affine coordinate when restricted to the projective line E. The notational abuse of equation (12.1) will be useful later on when defining coordinates on iterated blow-ups.

The coordinates (x, w_1) miss one point, the point $((0,0), [0,1])$ corresponding to $\ell_0 = y$-axis (and so $w_1 = \infty$). To cover this missing point use (y, v_1) for which $[v_1, 1]$ are the affine coordinates, and for which the the blow-down map is

$$\beta : (y, v_1) \mapsto (yv_1, y) = (x, y).$$

By the same abuse of notation, we write $v_1 = x/y$ in this case.

Blowing up the curve. If c is a curve with singular point at the origin, its blow-up is the curve $Bl^1(c) = \beta^{-1}(c) \subset Bl_0(\mathbb{C}^2)$. It consists of two components, the "proper transform" which is the closure of $\beta^{-1}(c \setminus \{0\})$, and the exceptional fiber E. If c is algebraic, its blow-up is algebraic. If c is analytic its blow-up is analytic.

Iterated blow-up. To resolve c we typically need to blow-up more than once. In order to blow-up a second time, we realize that $Bl_0(\mathbb{C}^2)$ is itself an analytic surface, and define the blow-up operation works on any analytic surface. So, let us define the blow-up $Bl_p(S)$ of an analytic surface S at a point $p \in S$. Choose coordinates (x, y) centered at p, coordinatizing a neighborhood U of p and so identifying U with a neighborhood V of 0 in $\mathbb{C}^2$. Using these coordinates, we identify $Bl_p(S)$ over p with the open set $\beta^{-1}(V) \subset Bl_0(\mathbb{C}^2)$. In these coordinates the blow-down map β

takes the same form as it did in the plane. The exceptional curve is $E = \beta^{-1}(p)$ and is a $\mathbb{P}^1 \subset Bl_p(S)$. Away from p, we declare $Bl_p(S) \to S$ to be an analytic diffeomorphism, and this endows $Bl_p(S)$ with the structure of an analytic surface. If $c \subset S$ is an analytic curve with singularity p then its blow-up at p is $\beta^{-1}(c)$, which again splits up into two parts, the proper transform, and the exceptional curve.

We are now able to iterate the blow-up process. Suppose that the first blow-up $B(c)$ of the unibranched curve singularity c is still singular. Then $B(c)$ will have a single singular point p_1 which must be the intersection point of the proper transform $\tilde{c}$ with the exceptional curve $E = E_1$. We blow-up $X_1 = Bl_0(\mathbb{C}^2)$ at p_1 so as to form a new surface $X_2 = Bl_{p_1}Bl_0(\mathbb{C}^2)$, and a new blown up curve $B^2(c) = B(B(c)) \subset X_2$ which consists of the new proper transform, still denoted $\tilde{c}$, and two exceptional curves, the new one, written E_2, and the proper transform of the old one, typically written $\tilde{E}_1$. If this configuration is still deemed singular, we keep going. At the kth iteration of the process we have a curve $Bl^k(c)$ in an analytic surface $X_k = Bl_{p_k}(Bl_{p_{k-1}} \ldots Bl_0(\mathbb{C}^2)\ldots)$, with $p_i \in E_i$. The kth blow-up of the curve has $k+1$ components (in the Zariski sense): $B^k(c) = \tilde{c}^k \cup \tilde{E}_1 \cup \ldots \cup \tilde{E}_{k-1} \cup E_k$, with $\tilde{c}^k$ denoting the proper transform of the original curve at the kth step, and $\tilde{E}_j, j < k$ denoting the proper transform of the exceptional curve E_j arising from the earlier level j. We stop the process when this collection of curves is normal in the sense of definition 5.1. (We take the distribution D of that definition, of course, to be the whole tangent bundle of X_k.)

13. The blow-up RVT code

A point $p_2 \in E_2 \subset X_2$ of the exceptional fiber of the second blow-up is a line through some point $p_1 \in E_1$. If that line is tangent to E_1 we say the point p_2 is a vertical point. Otherwise we say p_2 is a regular point. More generally, a point $p \in E_k \subset X_k$ represents a line (in some coordinate system) through a point $\beta(p) \in E_{k-1}$.

Definition 13.1 (cf. Definition 7.1)**.**
- The line through $p_k \in E_k \subset X_k$, $k \geq 1$, which is tangent to E_k is called the vertical line.
- A line through p_k which is tangent to some $\tilde{E}_j, j < k$ is called a tangency line.
- A line which is either vertical or tangency is called a critical line.
- A line which is not critical is called regular.
- A point $p \in E_k \subset X_k$, $k \geq 2$, is called a regular, vertical, or tangency point depending on whether or not the line it represents, passing through $\beta(p) \in X_{k-1}$, is regular, vertical, or tangency.
- A point which is either vertical or tangency is called critical.
- All points of $E_1 \subset X_1$ are deemed to be regular.

Proposition 13.2 (cf. Proposition 7.2)**.** *Let $p \in E_k \subset X_k$, $k \geq 1$. If p is a regular point then there is exactly one critical line passing through p and it is the vertical line E_k. If p is a critical point then there are exactly two critical lines passing through p: the vertical line E_k and a tangency line $\tilde{E}_j$, $j < k$. These two lines intersect transversally at p.*

We can now speak of the blow-up RVT code of a point $p \in E_k \subset X_k$, $k \geq 2$. We look at its blowdowns $p_j \in E_j \subset X_j$ and mark them R, V, or T according to whether or not they are regular, vertical or tangency points. The RVT code of $p \in E_k$ is thus a k-tuple $\alpha_1\alpha_2\ldots\alpha_k$ of letters α_i. Proposition 7.2 asserts that this word is subject to the same rules of grammar as the RVT codes introduced earlier for the Monster. According to our rules, $\alpha_1 = \mathrm{R}$.

Proposition 13.3 (key results on the Blow-up RVT code)**.** *Let $c(t)$ be unibranched plane curve germ singularity and $B^k(c)$ its resolution. Then the blow-up* RVT *code of the final point of the proper transform $\tilde{c}(0) \in B^k(c)$ determines the incidence relations in $B^k(c)$ according to five incidence rules expressed in Theorem 9.7.*

Proof. The proof follows very closely proof of Theorem 9.7.

We begin by observing that the obvious analogues of Propositions 8.1 and 8.2 hold, with very similar proofs. These analogues will be our basic tools.

We will only prove two out of the five incidence rules here: (1) that $\alpha_{i-1}\alpha_i = \mathrm{RV}$ implies that E_i is not incident to any $E_j, j < i$, and (2) that $\alpha_{i-1}\alpha_i = \mathrm{CV}$, where $\mathrm{C} \in \{\mathrm{V}, \mathrm{T}\}$, implies that E_i is incident to $E_{a(i-1)}$. The reader will easily see from the proofs of these two rules how to alter the proofs of the remaining three rules from Theorem 9.7 to give proofs of their blow-up analogues.

The case $\alpha_{i-1}\alpha_i = \mathrm{RV}$. Since $\alpha_i = \mathrm{V}$, the proper transform at level $i-1$ is tangent to the vertical curve E_{i-1} at that level. Upon blow-up this means that the proper transform at level i intersects $\tilde{E}_{i-1}$ in addition to E_i. Call the intersection point p_i. Having a triple intersection at p_i we must blow-up again in order to resolve, with center p_i. When we blow-up again, $\tilde{E}_{i-1}$ and $\tilde{E}_i$ become separated and no longer intersect. So E_i is not incident to E_{i-1}. Since $\alpha_{i-1} = \mathrm{R}$, , according to the blow-up analogue of (c) of Proposition 8.1, the only exceptional curve intersecting E_i is E_{i-1}, so that E_i is not incident to any E_j with $j < i$.

The case $\alpha_{i-1}\alpha_i = \mathrm{CV}$. Continuing as above, except with α_{i-1} critical we have that E_i intersects two critical curves at level i, namely , E_{i-1} and $E_{\alpha(i-1)}$. Only one of these, E_{i-1} also intersects the proper transform at level i, since the two critical directions at level $i-1$ are distinct. So E_i intersects $E_{\alpha(i-1)}$ in this case. □

14. Computing blow-up RVT codes

We closely follow section 11 above. As with the RVT code, there are two ways to compute the blow-up RVT code of a plane curve germ. One way uses the Puiseux characteristic of the curve. It would have appeared in [16], except that the blow-up RVT code does not appear in Wall. The other way uses a special atlas of coordinates on the blow-ups, the blow-up coordinates. As with our earlier discussion, we can circumvent direct use of these coordinates by iterating an operator

$$\mathcal{B} : \mathcal{C} \to \mathcal{C}$$

Here $\mathcal{C}$ is the same space on which the operator $\mathcal{P}$ of sect.11 was defined. We recall $\mathcal{C} = \big\{(x(t), y(t) : x(0) = y(0) = 0,\ \mathrm{ord}\, x(t) \leq \mathrm{ord}\, y(t)\big\}$. For $(x(t), y(t)) \in \mathcal{C}$ set

$$w(t) = \frac{y(t)}{x(t)} - \Big(\frac{y(t)}{x(t)}\Big)(0).$$

Then we define $\mathcal{B}$ by

$$\mathcal{B}((x(t), y(t))) = \begin{cases} (x(t), w(t)) \text{ if } \mathrm{ord}\, x(t) \leq \mathrm{ord}\, z(t) \\ (w(t), x(t)) \text{ if } \mathrm{ord}\, x(t) > \mathrm{ord}\, z(t). \end{cases}$$

Iterating $\mathcal{B}$ defines a sequence of curve germs and function germs:

$$\mathcal{B}^i(c) = (\tilde{x}_i(t), \tilde{y}_i(t)), \qquad w_i(t) = \frac{\tilde{y}_i(t)}{\tilde{x}_i(t)} - \Big(\frac{\tilde{y}_i(t)}{\tilde{x}_i(t)}\Big)(0), \quad i = 1, 2, \dots.$$

The following proposition is the blow-up analogue of Propositions 11.2 and 11.3.

Proposition 14.1 (cf. Propositions 11.2 and 11.3)**.** *Let $(x_0(t), y_0(t)) \in \mathcal{C}$ be a well-parameterized curve germ with blow-up* RVT *code* $\mathrm{R}\tilde{\ell}_1 \dots \tilde{\ell}_{\tilde{\mathrm{k}}-1}$ *and let $(\tilde{x}_i, \tilde{y}_i) = \mathcal{B}^i(x_0, y_0)$, and w_i be as above. Then*

$$\tilde{\ell}_{i+1} = \mathrm{R} \implies \tilde{\ell}_{i+2} = \begin{cases} \mathrm{R} \text{ if } \operatorname{ord} \tilde{x}_i(t) \le \operatorname{ord} w_i(t) \\ \mathrm{V} \text{ if } \operatorname{ord} \tilde{x}_i(t) > \operatorname{ord} w_i(t) \end{cases}$$

$$\tilde{\ell}_{i+1} = \mathrm{V} \textit{ or } \mathrm{T} \implies \tilde{\ell}_{i+2} = \begin{cases} \mathrm{T} \text{ if } \operatorname{ord} \tilde{x}_i(t) < \operatorname{ord} w_i(t) \\ \mathrm{R} \text{ if } \operatorname{ord} \tilde{x}_i(t) = \operatorname{ord} w_i(t) \\ \mathrm{V} \text{ if } \operatorname{ord} \tilde{x}_i(t) > \operatorname{ord} w_i(t). \end{cases}$$

The integer $\tilde{k}$, being the length of the blow-up RVT *code plus one, is first integer such that* $\operatorname{ord}(\tilde{x}_{\tilde{k}}) = \operatorname{ord}(\tilde{y}_{\tilde{k}}) = 1$*, and the curve is resolved upon $\tilde{k} + 1$ blow-ups.*

We will not prove the theorem, but provide a sketch of the idea. The proof relies on blow-up coordinates, which are analogues of the coordinates on the Monster, see [13]. The ith blow-up coordinates are associated to a point $p_i \in E_i$, taken to be the origin of the ith proper transform of our curve. The coordinates can be written $(w_{A(i)}, w_i)$, with w_i being an affine coordinate on the exceptional curve E_i which itself is defined by the equation $w_{A(i)} = 0$, The index $A(i) < i$ is an earlier appearing index, so that at some earlier stage of blow-up $w_{A(i)}$ either represented the x-coordinate of the curve, or an affine coordinate on $E_{A(i)}$. One key ingredient of the proof is to verify that $\mathcal{B}^i(x_0, y_0)$ equals either $(w_{A(i)}, w_i)$ or $(w_i, w_{A(i)})$.

15. Proof of the main theorem

Our main theorem, Theorem 6.2, asserts that the resolution diagrams for prolongation and blow-up are the same diagrams. Theorems 9.7 describes an algorithm which computes the graph for resolution by prolongation out of the curve's RVT code. Proposition 13.3 asserts that *the same algorithm* computes the diagram for resolution by blow-up from the curve's blow-up RVT code. Theorem 15.1 below asserts that these two codes are equal. Combining these results we see that we will have proved our main theorem once we have proved Theorem 15.1 .

Theorem 15.1. *The prolongation*-RVT-*code of an analytic plane curve germ coincides with its blow-up*-RVT-*code.*

Proof of Theorem 15.1. Let $(\mathrm{R}, \mathrm{l}_2, \mathrm{l}_3, ...)$ be the prolongation-RVT-code of a curve $c \in C$ and let $(\mathrm{R}, \widetilde{\mathrm{l}}_2, \widetilde{\mathrm{l}}_3, ...)$ be the blow-up-RVT-code of the same curve c. We have to prove that $l_i = \widetilde{l}_i$. We will prove this by induction by i. It is obvious that $\ell_2 = \widetilde{\ell}_2$ so we have the base of the induction and we will prove the implication

$$\ell_2 = \widetilde{\ell}_2, \ \dots, \ell_{m+1} = \widetilde{\ell}_{m+1} \implies \ell_{m+2} = \widetilde{\ell}_{m+2}, \quad m \ge 1. \tag{15.1}$$

Recall the notation $\mathcal{P}^i(x_0, y_0) = (x_i, y_i)$ and $\mathcal{B}^i(x_0, y_0) = (\tilde{x}_i, \tilde{y}_i)$. We will use Propositions 11.2, 14.1 and the following:

Proposition 15.2. *The curves* $(x_i(t), y_i(t))$ *and* $(\widetilde{x}_i(t), \widetilde{y}_i(t))$ *have the same Puiseux characteristics.*

To prove Proposition 15.2 it suffices to prove the following lemma (Proposition 15.2 is obtained by iterating this lemma).

Lemma 15.3. *If* $c, \widetilde{c} \in \mathcal{C}$ *are curves with the same Puiseux characteristics then the curves* $\mathcal{P}(c)$ *and* $\mathcal{B}(\widetilde{c})$ *also have the same Puiseux characteristics.*

Proof. Since the Puiseux characteristic of a curve coincides with the Puiseux characteristic of any its reparameterization and since the operators $\mathcal{P}$ and $\mathcal{B}$ respect reparameterization, we may assume

$$c = \big(t^{\lambda_0},\ f(t)\big), \quad \widetilde{c} = \big(t^{\lambda_0},\ g(t)\big),$$

where $\operatorname{ord} f(t) \geq \lambda_0$ and $\operatorname{ord} g(t) \geq \lambda_0$. Consider the curve

$$\widehat{c} = \big(t^{\lambda_0}, \ \ tf'(t)/\lambda_0\big).$$

Obviously c and $\widehat{c}$ have the same Puiseux characteristics. Therefore $\widetilde{c}$ and $\widehat{c}$ have the same Puiseux characteristics. The blow-ups of two curves with the same Puiseux characteristics also have the same Puiseux characteristics (this is proved in Wall's book [16]). Therefore the curves $\mathcal{B}(\widetilde{c})$ and $\mathcal{B}(\widehat{c})$ have the same Puiseux characteristics. It remains to note that $\mathcal{P}(c) = \mathcal{B}(\widehat{c})$. □

Now we will prove (15.1) using Propositions 11.2, 14.1 and 15.2. We will consider all possible cases: $\ell_{m+1} = \mathrm{R}$, $\ell_{\mathrm{m}+1} = \mathrm{V}$, $\ell_{\mathrm{m}+1} = \mathrm{T}$. *All the time we will use Propositions 11.2 and 15.2, without mentioning this.*

1. The case $\ell_{m+1} = \widetilde{\ell}_{m+1} = \mathrm{R}$. Let $[\lambda_0; \lambda_1, ...]$ be the Puiseux characteristic of the curves

$$(x_m(t), y_m(t)), \quad (\widetilde{x}_m(t), \widetilde{y}_m(t)) \tag{15.2}$$

1.1. Subcase: $\lambda_1 < 2\lambda_0$. In this case the curves (15.2) have, up to reparameterization, the form

$$\big(t^{\lambda_0},\ bt^{\lambda_0} + at^{\lambda_1} + \text{h.o.t}\big), \quad \big(t^{\lambda_0},\ \widetilde{b}t^{\lambda_0} + \widetilde{a}t^{\lambda_1} + \text{h.o.t}\big), \quad a, \widetilde{a} \neq 0, \quad \lambda_1 < 2\lambda_0$$

and in this case $\operatorname{ord}(z_m) = \operatorname{ord}(\widetilde{z}_m) = \lambda_1 - \lambda_0 < \lambda_0$ so that $\ell_{m+2} = \widetilde{\ell}_{m+2} = \mathrm{V}$.

1.2. Subcase: $\lambda_1 > 2\lambda_0$. Take integer $r \geq 2$ such that

$$r\lambda_0 < \lambda_1 < (r+1)\lambda_0, \quad r \geq 2. \tag{15.3}$$

The curves (15.2) have, up to reparameterization, the form

$$\big(t^{\lambda_0}, \ \ b_1 t^{\lambda_0} + b_2 t^{2\lambda_0} + \cdots + b_r t^{r\lambda_0} + at^{\lambda_1} + \text{h.o.t}\big), \quad a \neq 0 \tag{15.4}$$

$$\big(t^{\lambda_0}, \ \ \widetilde{b}_1 t^{\lambda_0} + \widetilde{b}_2 t^{2\lambda_0} + \cdots + \widetilde{b}_r t^{r\lambda_0} + \widetilde{a}t^{\lambda_1} + \text{h.o.t}\big), \quad \widetilde{a} \neq 0. \tag{15.5}$$

Some part (or all) of coefficients b_i might be zero. Independently some (or all) of coefficients $\widetilde{b}_i$ might be zero. In *any* case $\ell_{m+2} = \widetilde{\ell}_{m+2} = \mathrm{R}$.

We are done with the case $\ell_{m+1} = \widetilde{\ell}_{m+1} = \mathrm{R}$. The cases $\ell_{m+1} = \widetilde{\ell}_{m+1} = \mathrm{V}$ and $\ell_{m+1} = \widetilde{\ell}_{m+1} = \mathrm{T}$ require the following lemma.

Lemma 15.4. *Let* $[\lambda_0; \lambda_1, ...]$ *be the Puiseux characteristic of the curve* $(x_{i-1}(t), y_{i-1}(t))$. *If* $\ell_{i+1} = \mathrm{V}$ *then* $\lambda_1 < 2\lambda_0$.

Proof. Assume, to get contradiction, that $\lambda_1 > 2\lambda_0$. Take integer $r \geq 2$ satisfying (15.3). Up to reparameterization the curves $(x_{i-1}(t), y_{i-1}(t))$ has the form (15.4). Then by Proposition 11.2 ℓ_{i+1} is either R or T, but not V. Contradiction. □

2. The case $\ell_{m+1} = \widetilde{\ell}_{m+1} = \mathrm{V}$. Let $[\lambda_0; \lambda_1, ...]$ be the Puiseux characteristic of the curves

$$(x_{m-1}(t), y_{m-1}(t)), \quad (\widetilde{x}_{m-1}(t), \widetilde{y}_{m-1}(t)). \tag{15.6}$$

By Lemma 15.4 one has $\lambda_1 < 2\lambda_0$ and consequently the curves (15.6) have, up to reparameterization, the form

$$\big(t^{\lambda_0},\ \ at^{\lambda_1} + \text{h.o.t.}\big),\ \ \big(t^{\lambda_0},\ \ \widetilde{a}t^{\lambda_1} + \text{h.o.t.}\big),\ \ a \neq 0,\ \widetilde{a} \neq 0,\ \lambda_0 < \lambda_1 < 2\lambda_0. \tag{15.7}$$

It follows that the curves $(x_m(t), y_m(t))$ and $(\widetilde{x}_m(t), \widetilde{y}_m(t))$ have, up to reparameterization, the form

$$\big(ct^{\lambda_1-\lambda_0} + \text{h.o.t.},\ \ t^{\lambda_0}\big),\ \ \big(\widetilde{c}t^{\lambda_1-\lambda_0} + \text{h.o.t.},\ \ t^{\lambda_0}\big),\ \ c \neq 0,\ \widetilde{c} \neq 0. \tag{15.8}$$

By Propositions 11.2 and 14.1 one has:

$$\ell_{m+2} = \widetilde{\ell}_{m+2} = \begin{cases} \text{R if } \lambda_1 = (3/2)\lambda_0 \\ \text{V if } \lambda_1 > (3/2)\lambda_0 \\ \text{T if } \lambda_1 < (3/2)\lambda_0 \end{cases}.$$

3. The case $\ell_{m+1} = \widetilde{\ell}_{m+1} = \mathrm{T}$. In this case there exists $q \leq m-1$ such that

$$\ell_{q+1} = \widetilde{\ell}_{q+1} = \mathrm{V},\ \ \ell_{\mathrm{q+2}} = \widetilde{\ell}_{\mathrm{q+2}} = \cdots = \ell_{\mathrm{m+1}} = \widetilde{\ell}_{\mathrm{m+1}} = \mathrm{T}. \tag{15.9}$$

Let $[\lambda_0; \lambda_1, ...]$ be the Puiseux characteristic of the curves

$$(x_{q-1}(t), y_{q-1}(t)), \quad (\widetilde{x}_{q-1}(t), \widetilde{y}_{q-1}(t)). \tag{15.10}$$

By Lemma 15.4 one has $\lambda_1 < 2\lambda_0$ and consequently the curves (15.10) have, up to reparameterization, the form (15.7). It follows that the curves $(x_q(t), y_q(t))$ and $(\widetilde{x}_q(t), \widetilde{y}_q(t))$ have, up to reparameterization, the form (15.8). Since $\ell_{q+2} = \widetilde{\ell}_{q+2} =$ T then by Propositions 11.2 and 14.1 one has $\lambda_1 < (3/2)\lambda_0$. Therefore the curves $(x_{q+1}(t), y_{q+1}(t))$ and $(\widetilde{x}_{q+1}(t), \widetilde{y}_{q+1}(t))$ have, up to reparameterization, the form

$$\begin{aligned} &\big(ct^{\lambda_1-\lambda_0} + \text{h.o.t.},\ \ c_1t^{2\lambda_0-\lambda_1} + \text{h.o.t.}\big),\ \ c, c_1 \neq 0 \\ &\big(\widetilde{c}t^{\lambda_1-\lambda_0} + \text{h.o.t.},\ \ \widetilde{c}_1t^{2\lambda_0-\lambda_1}\big),\ \ \widetilde{c}, \widetilde{c}_1 \neq 0 \\ &\lambda_0 < \lambda_1 < (3/2)\lambda_0. \end{aligned} \tag{15.11}$$

Since $\ell_{q+3} = \widetilde{\ell}_{q+3} =$ T then Proposition 11.2 implies that λ_0 and λ_1 satisfy $\lambda_1 < (4/3)\lambda_0$. Continuing in this manner, one proves by induction that as long as $\ell_{q+2+i} = \widetilde{\ell}_{q+2+i} =$ T we have both the curves $(x_{q+i}(t), y_{q+i}(t))$ and $(\widetilde{x}_{q+i}(t), \widetilde{y}_{q+i}(t))$ have the form

$$\begin{aligned} &\big(ct^{\lambda_1-\lambda_0} + \text{h.o.t.},\ \ c_it^{(i+1)\lambda_0-i\lambda_1} + \text{h.o.t.}\big),\ \ c, c_i \neq 0 \\ &\lambda_0 < \lambda_1 < \frac{i+3}{i+2}\lambda_0. \end{aligned} \tag{15.12}$$

Continuing on up to the case $q+2+i = m+1$ so that $i = m-q-1$ we find that

$$\lambda_0 < \lambda_1 < \frac{m-q+2}{m-q+1}\lambda_0, \tag{15.13}$$

while both the curves $(x_{m-1}(t), y_{m-1}(t))$ and $(\tilde{x}_{m-1}(t), \widetilde{y}_{m-1}(t))$ must have the form

$$\big(ct^{\lambda_1-\lambda_0} + \text{h.o.t.},\quad c^* t^{(m-q)\lambda_0-(m-q-1)\lambda_1} + \text{h.o.t.}\big),\quad c, c^* \neq 0 \tag{15.14}$$

The next step yields that both the curves $(x_m(t), y_m(t))$ and $(\tilde{x}_m(t), \widetilde{y}_m(t))$ must have the form

$$\big(ct^{\lambda_1-\lambda_0} + \text{h.o.t.},\quad c^* t^{(m-q+1)\lambda_0-(m-q)\lambda_1} + \text{h.o.t.}\big),\quad c, c^* \neq 0 \tag{15.15}$$

and that both $z_m(t), \tilde{z}_m(t)$ have the form of the form $ct^{(m-q+2)\lambda_0-(m-q+1)\lambda_1}$ +h.o.t.

We arrive at the conclusion: in the case (15.9) the first entries λ_0, λ_1 in the Puiseux characteristic of the curves $(x_{q-1}(t), y_{q-1}(t))$ and $(\widetilde{x}_{q-1}(t), \widetilde{y}_{q-1}(t))$ satisfy (15.13), the curves $(x_m(t), y_m(t))$ and $(\widetilde{x}_m(t), \widetilde{y}_m(t))$ have the form (15.15) and consequently (once again by Propositions 11.2 and 14.1) one has

$$\ell_{m+2} = \widetilde{\ell}_{m+2} = \begin{cases} \text{R if } \lambda_1 = \frac{m-q+3}{m-q+2}\lambda_0 \\ \text{V if } \lambda_1 > \frac{m-q+3}{m-q+2}\lambda_0 \\ \text{T if } \lambda_1 < \frac{m-q+3}{m-q+2}\lambda_0. \end{cases}$$

Theorem 15.1 is proved.

16. Final Words. Open problems

A tragedy befell the writing of this story, a tragedy which befalls most mathematical papers. In the interests of clarity and brevity, we have covered up our tracks: we have hidden or destroyed scenic trails which led us to the story. The two trails we have hidden have separate beginnings. One trail begins with deformations. The other begins with Puiseux characteristics.

We had been looking at the prolongations of the family whose curves form the versal deformation of the cubic cusp. To our dismay, we discovered that this family of prolongations was discontinuous in the deformation parameter. The only way to recover a vestige of continuity was to add the vertical fiber in to the prolongation of the cusp. The similarity with blow-up became apparent and led us to this story. Details of this trail can be found in Chapter 8 of our book [13].

The other trail we have hidden concerns the combinatorial interaction between Puiseux characteristics and RVT codes. In Theorem 4.3.8 of the book [16] Wall lists 8 separate types of combinatorial invariants associated to a plane curve singularity, and he proves that all eight invariants are equivalent. The first of these invariants is the Puiseux characteristic. In [13] we proved, that the RVT code can be added to Wall's list, forming a 9th equivalent invariant. An earlier strategies for proving the main theorem of the present paper was to construct two bijections between Puiseux characteristics and RVT codes, one map for prolongation, the other for blow-up, and then to show that the maps are the same. This strategy ultimately became confusing and we deleted it. But in the process we have deleted some of the beauty around Puiseux characteristics.

The first hidden trail just described suggest the following problem.

A. Investigate the extent to which deformation and prolongation of curves commute.

And mathematician's irrepressible urge to generalize forces us to pose:

B. Develop a good theory of prolongation of hypersurface singularities.

Curves in the plane are the case $n = 2$ of hypersurfaces in n-space. Nash blow-up is defined for analytic singularities of arbitrary codimension $c = n - k$ in n-space. The case of hypersurfaces should be particularly interesting from the point of view of symmetry groups, due to a theorem of Yamaguchi [17]. The first prolongation of a singular hypersurface will be a Legendrian subvariety of a a contact manifold. The symmetry group of this contact manifold is much larger than the group of diffeomorphisms of the ambient n- space (the step 0 case) within which the original hypersurface lives. On the other hand, Yamaguchi's theorem asserts that for all other codimensions c, $1 < c < n$. the group of symmetries of the "first prolongation distribution" coincides with the diffeomorphism group. See [17].

References

[1] V. I. Arnold, *Topological Invariants of Plane Curves and Caustics*, University Lecture Series, v. 5, AMS, 2000.

[2] E. Brieskorn, H. Knorrer, **Plane Algebraic Curves**, Birkhauser, 1986.

[3] R. Bryant, L. Hsu, *Rigidity of integral curves of rank two distributions.* Invent. Math., **114** (1993) 435 - 461.

[4] E. Cartan, *Sur l'equivalence absolue de certains systemes d'equations differentielles.* C.R. Acad. Sci., **158** (1914), p. 326-330; Ouevres completes, partie II, 2, 1129-1132, Gauthier Villars, 1953.

[5] E. Cartan, *Sur l'equivalence absolue de certains systemes d'equations differentielles et sur certaines familles de courbes.* Ann. Ec. Norm. Sup. **42** (1914), 12-48; Ouevres completes, partie II, 2, 1133-1168, Gauthier Villars, 1953.

[6] A. Chenciner, **Courbes Algebiques Planes** 2nd ed. , Springer, 2007.

[7] Theo de Jong, G. Pfister, **Local Analytic Geometry** in Advanced Lectures in Mathematics, Vieweg, 2000.

[8] R. Gompf , A. Stipsicz, **4-manifolds and Kirby calculus** AMS, 1999.

[9] J. Harris, **Algebraic Geometry, a first course**, esp. p. 221, GTM Vol. 133,Springer,1992

[10] J. Kollar, electronic communication, 2008.

[11] J. Kollar, **Lectures on Resolution of Singularities**, Annals of Mathematics Studies, Princeton Univ. Press, 2007.

[12] R.Montgomery, M.Zhitomirskii, *Geometric Approach to Goursat Flags.* Annales de L'Institut Henri Poincare. Analyse non Lineaire **18** (2001), no. 4, 459-493.

[13] R.Montgomery, M.Zhitomirskii, **Points and curves in the Monster Tower**, submitted to Memoirs of the AMS, 2008

[14] A. Nobile, *Some Properties of the Nash Blowing-up*, Pac. J. Math, Vol. 60, No. 1, 1975.

[15] M. Spivakovsky, *Sandwiched Singularities and Desingularization of Surfaces by Normalized Nash Transformations*, Annals of Mathematics, 2nd Ser., Vol. 131, No. 3. (May, 1990), pp. 411-491.

[16] C.T.C. Wall, **Singular Points of Plane Curves**. London Math. Soc. Student Text **63**, Cambridge Univ. Press, Cambridge, 2004.

[17] K. Yamaguchi, *Contact geometry of higher order.* Japan J. Math., **8** (1982), 109-176

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