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Gromov, Mikhael
Carnot-Carathéodory spaces seen from within.
Sub-Riemannian geometry, 79–323, *Progr. Math.*, 144, *Birkhäuser*, *Basel*, 1996.
53C17 (53C23)

FEATURED REVIEW.Carnot-Carathéodory, or CC, metric consists of a subbundle H of the tangent bundle TV of a manifold V, together with a fiber-inner product on H. These metrics arise in the studies of: (1) limits of Riemannian and other metric spaces, (2) the Mostow rigidity phenomenon, (3) optimal control and (4) PDE, specifically, hypoelliptic operators. CC metrics are also known as sub-Riemannian metrics or singular Riemannian metrics. Gromov is the inventor of (1) [in *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, 415–419, Acad. Sci. Fennica, Helsinki, 1980; MR 81g:53029; *Structures métriques pour les variétés riemanniennes*, Edited by J. Lafontaine and P. Pansu, CEDIC, Paris, 1981; MR 85e:53051; Inst. Hautes Études Sci. Publ. Math. No. 53, (1981), 53–73; MR 83b:53041]. See [P. Pansu, Ann. of Math. (2) **129** (1989), no. 1, 1–60; MR 90e:53058] for (2). See [R. W. Brockett, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, 1357–1368, PWN, Warsaw, 1984; MR 86k:93068] for (3), which can also serve as an introduction to CC geometries. The seminal work in (4) is [L. Hörmander, Acta Math. **119** (1967), 147–171; MR **36** #5526], with [L. P. Rothschild and E. M. Stein, Acta Math. **137** (1976), no. 3-4, 247–320; MR **55** #9171] being another good early source. For a friendly philosophical overview of CC spaces I recommend the paper by R. S. Strichartz [Math. Intelligencer **9** (1987), no. 3, 56–64; MR 88h:35002]. Another source with a review-like nature by Strichartz is [J. Differential Geom. **24** (1986), no. 2, 221–263; MR 88b:53055], but note the corrections [J. Differential Geom. **30** (1989), no. 2, 595–596; MR 90f:53081]. The book under review is another book-within-a-conference proceedings by Gromov. (For earlier work by Gromov, see [in *Geometric group theory, Vol. 2 (Sussex, 1991)*, 1–295, Cambridge Univ. Press, Cambridge, 1993; MR 95m:20041].)

Call a curve in V horizontal if it is tangent to H. We can measure the length of such a curve by using the inner product on H. We obtain a distance function dist on V by declaring that $\mathrm{dist}(x, y)$

be the infimum of the lengths of the horizontal curves joining x to y. In the "extrinsic" point of view, one uses strongly H sitting within the tangent bundle, and all the consequent geometry relating the twisting of H and of its inner product. The "intrinsic" point of view, the "seen from within" of this book's title, refers to viewing V simply as a metric space and seeing how far one can get with the metric alone. In this way the book is a natural continuation of the author's earlier book [op. cit.; MR 85e:53051], where this viewpoint is adopted towards Riemannian geometry. (The latter reference was recently expanded and translated into English [*Metric structures for Riemannian and non-Riemannian spaces*, Translated from the French by Sean Michael Bates, Progr. Math., 152, Birkhäuser Boston, Boston, MA, 1999; MR 2000d:53065].)

A basic intrinsic question addressed by the book is: Can we recover the distribution H from dist alone? Other questions addressed in the book are: What is the nature of a CC minimal surface? What is the nature of CC isoperimetric inequalities? What are the Hausdorff dimensions of submanifolds of V? What form do the Sobolev inequalities take in the CC world?

The fundamental theorem of the subject is Chow's theorem, also called the Chow-Rashevskiĭ theorem. It is the subject of Chapter 1. This theorem asserts that if H generates TV under iterated Lie bracket then the distance $\mathrm{dist}(x, y)$ is finite for y in a neighborhood of x. In other words, if y can be joined to x by some path, then it can be joined to it by a horizontal path. It is a kind of opposite to the Frobenius integrability theorem. Gromov's treatment of Chow is not so different from other treatments, with one exception. He provides a proof that the horizontal connecting paths can be taken to be smooth. All proofs rely somehow on the "quadrilateral approximation" $\exp(-tY)\exp(-tX)\exp(tY)\exp(tX)(x) = x + t^2[X, Y](x) + O(t^4)$ to the Lie bracket of vector fields X and Y, and without further work yield piecewise smooth, as opposed to smooth, horizontal connectors.

The Chow connectivity theorem becomes quantitative upon introducing the sheaves of vector fields H_j defined by $H_1 = H$, $H_2 = [H, H] + H$ and $H_{j+1} = [H, H_j] + H_j$. In other words H_j is spanned by all s-fold Lie brackets of horizontal vector fields, with $s \leq j$. The assumption that bracket by H generates the tangent bundle is the requirement that $H_r = TV$ for r sufficiently large. Then $0 \subset H \subset H_2 \subset \cdots \subset H_r = TV$ forms a filtration of the tangent sheaf. Set $W_j(x) = H_j(x)/H_{j-1}(x)$, and $k_j(x) = \dim(W_j(x))$. Gromov calls a distribution equiregular if these $k_j(x)$'s are independent of the point x. Suppose this to be the case. Set $\mathrm{Gr}(H)(x) = H(x) \oplus W_2(x) \oplus \cdots \oplus W_r(x) \cong \mathbf{R}^{k_1} \oplus \mathbf{R}^{k_2} \oplus \cdots \oplus \mathbf{R}^{k_r}$. Choose norms $\|\cdot\|_i$ on the factors $\mathbf{R}^{k_i}$ and set $\mathrm{Box}(\varepsilon) = \{(x_1, x_2, \cdots, x_r) \in \mathbf{R}^{k_1} \oplus \mathbf{R}^{k_2} \oplus \cdots \oplus \mathbf{R}^{k_r}\colon \|x_i\|_i \leq \varepsilon^i\}$. The ball-box theorem asserts that there exist coordinates $x = (x_1, \cdots, x_r)$, $x_i \in \mathbf{R}^{k_i}$, centered at equiregular $q \in V$, together with positive constants c, C such that in these coordinates $\mathrm{Box}(c\varepsilon) \subset B_{\mathrm{dist}}(q, \varepsilon) \subset \mathrm{Box}(C\varepsilon)$. Versions of this can be found in the papers by Rothschild and Stein [op. cit.], A. Nagel, Stein and S. Wainger [Acta Math. **155** (1985), no. 1-2, 103–147; MR 86k:46049], and V. Ya. Gershkovich and A. M. Vershik [J. Geom. Phys. **5** (1988), no. 3, 407–452; MR 91j:58011]. Gromov gives his own proof. Perhaps the most careful version in existence is that of A. Bellaïche [in *Sub-Riemannian geometry*, 1–78, Progr. Math., 144, Birkhäuser, Basel, 1996; MR 98a:53108], which precedes Gromov's contribution to this conference proceedings.

The coordinate volume of $\mathrm{Box}(\varepsilon)$ is a constant times ε^N, where $N = \sum ik_i$. It follows from this and the ball-box theorem that the Hausdorff dimension of V with respect to dist is N. On the other hand, the topological dimension is $n = \sum k_i$. It follows that for a CC geometry ($H \neq TV$) the Hausdorff dimension is always greater than the topological dimension.

Another important theorem, closely related to the ball-box theorem, is Mitchell's theorem [J. W. Mitchell, J. Differential Geom. **21** (1985), no. 1, 35–45; MR 87d:53086]. To state it, one begins by observing that the Lie bracket of vector fields induces a Lie algebra structure on $\mathrm{Gr}(H)(x)$ in the equiregular case. This becomes a graded nilpotent Lie algebra. The exponential of this algebra is a simply connected nilpotent group G with a canonical CC structure: take $H(x)$ and left-translate it about the group. Such G's are called Carnot groups by Gromov's school, and homogeneous groups by Stein and his school [G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Princeton Univ. Press, Princeton, N.J., 1982; MR 84h:43027]. If (V, H) is a contact manifold then this G is the n-dimensional Heisenberg group, and is independent of x. In general, it depends on x. Mitchell's theorem asserts that the Gromov tangent cone to (V, dist) at x is this Carnot group. We find a proof of this in Chapter 1. For the most careful proof available, again see Bellaïche's article (cited above) preceding Gromov's. Bellaïche works even in the nonequiregular case.

CC geometries first crept into Gromov's work in [op. cit.; MR 83b:53041]. Essentially, he proved that the tangent cone at infinity of an infinite discrete group of polynomial growth is a Carnot group. In this way he was able to deduce that all such discrete groups are virtually nilpotent! This is also one of the first papers which uses the Gromov-Hausdorff topology.

Leaving Chapter 1 and the relatively well-known world of Chow, we come to Chapter 2, which concerns hypersurfaces in V with metric geometry induced by restricting the CC distance function. One of the first results is that if W is a compact subset of V of topological dimension $n-1$ then its Hausdorff dimension is at least $N-1$. A CC isoperimetric inequality is proved: $\mathrm{meas}_{N-1}(\partial D) \leq C\mathrm{meas}_N(D)^{(N-1)/N}$ for domains D lying within compact regions of V. Here meas_k denotes the Hausdorff k-dimensional measure, ∂D is assumed to be smooth, at least in one incarnation of this result, and C is a fixed constant. Gromov gives two proofs of this inequality. One involves horizontal flow tubes. Another proof is based on what he calls "Green forms": CC analogues of the spherical area form on $\mathbf{R}^3 \setminus 0$. A Green form is an $(n-1)$-form ω defined on V minus a point and singular at that point, which is closed and horizontal: $\omega|_H = 0$. Gromov proves the existence of such forms and uses them in a manner similar to the calibrations arising in minimal surfaces to prove his isoperimetric inequality. The CC Sobolev inequality $\int |f|^{N/(N-1)} \leq C(\int |df|_H)^{N/(N-1)}$ follows in a by-now standard way. Here $df|_H$ is the restriction of the differential to the horizontal space H, and the norm of this horizontal differential is defined using the CC inner product. The derivation of the Sobolev inequality from the isoperimetric inequality follows a well-known line of thought [see, e.g., N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and geometry on groups*, Cambridge Univ. Press, Cambridge, 1992; MR 95f:43008]. Various other Sobolev inequalities follow from this basic one. At a critical Sobolev exponent Gromov explores the bubbling phenomenon à la Uhlenbeck. Here we find "pre-bubbles" and "bridges", a long "trivial" convergence lemma, and a wild picture, figure 4, which I confess I do not really understand.

He continues the chapter with investigations of taut CC maps, Hölder estimates in the critical exponent case. The chapter ends with "remarks and corollaries" centering around relations to quasi-conformal maps, and hence indirectly to item (2) of our first paragraph above, and to ideas which stem from his paper on groups of polynomial growth [op. cit.; MR 83b:53041]. Developments here seem to set the stage for a future theory of CC minimal surfaces.

Chapter 3 concerns the case where H is a contact field on V. As Gromov says at this chapter's end: "Our study of (V, H) [is] local and in a sense perpendicular to the global contact explosion of the last decade." He investigates the Hölder and Lipschitz properties of maps $\mathbf{R}^k \to V$. In $\S 3.5$ Gromov says: "Nobody knows yet (except possibly W. Thurston) whether every (closed horizontal) surface S in the Heisenberg group H^n for $n \geq 7$ bounds something three-dimensional of the volume (i.e. the three-dimensional Hausdorff measure) satisfying $\mathrm{volume} \leq C(\mathrm{Area})^{3/2}$." Such lapses into friendly conversational style, as if Thurston were standing in a neighboring room sipping his tea, lighten up the often heavy reading. Gromov goes on to describe Rumin's complex, a de Rham type complex adapted to the contact setting with interesting properties near the Legendrian dimension $(n-1)/2$ [see M. Rumin, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), no. 6, 401–404; MR 91a:58004]. I did not get a clear feeling of where Gromov was going in this chapter.

Of all the chapters, Chapter 4 captured my interest the most. Let $V' \subset V$ be a submanifold. Set $H'_i = H_i \cap TV'$. Set $k'_i = \mathrm{rank}(H'_i / H'_{i-1})$, and call the submanifold equiregular if these dimensions are constant along the submanifold. We find the formula $N' = \sum i k'_i$ for the Hausdorff dimension of $(V', \mathrm{dist}|_{V'})$, an equiregular submanifold. Gromov turns this formula around: fix N' (or fix various of the k'_i) and view this relation as a PDE for V', or more accurately, a PDR (R for "relation"). He investigates this PDR using tools from his earlier impenetrable masterpiece [*Partial differential relations*, Springer, Berlin, 1986; MR 90a:58201]. Sections 4.1 to 4.3 concern, in the main, horizontal submanifolds of a given dimension, which is the case $N' = \dim(V') = k'_1$. The curvature of a distribution is defined to be the map $\Omega\colon \Lambda^2 H \to TV/H$ sending $X \wedge Y$ to $[X, Y] \bmod H {:=} \Omega(X, Y)$. (W. P. Thurston calls this curvature the "torsion" of the distribution [*Three-dimensional geometry and topology. Vol. 1*, Edited by Silvio Levy, Princeton Univ. Press, Princeton, NJ, 1997; MR 97m:57016].) Fix a local coframe $\eta_1, \cdots, \eta_{n-k}$ of one-forms for the subbundle of T^*V which annihilates H. Viewed as an $m = (n-k)$-vector-valued form, the η_i provide an identification of TV/H with $\mathbf{R}^m$ and the restrictions of the differentials $d\eta_i$ to $H \times H$ provide a realization of this curvature. Horizontality of a submanifold is expressed by the relation $f^*\eta = 0$ where f is the immersion. One can linearize these equations by varying f and using Cartan's formula for the Lie derivative. (Here as in many places in the book we find notational ambiguities which might throw the inexperienced reader into confusion: see the last line on p. 249.) The linearization at a point p of the alleged manifold, with alleged horizontal tangent space S, defines a linear map $H_p \to \mathrm{Hom}(S, (TQ/H)_p)$. Gromov defines an "$\Omega$-regular subspace" of $S \subset H_p$ to be one for which this map is onto. Note that Ω-regularity can only hold when the corank of H is not too big. Gromov then asserts the validity of various h-principles for families of horizontal submanifolds in which Ω-regularity is in force. The example of rigid curves arising in rank-2 distributions serves as a warning to any sweeping validity of horizontal h-principle theorems. (Their tangents are not Ω-regular subspaces.) These are horizontal curves which admit no C^1 horizontal variations with fixed endpoints. They are a generic phenomenon in rank 2. These curves have their roots in developments in the calculus of variations at the beginning of this century [see R. L. Bryant and L. Hsu, Invent. Math. **114** (1993), no. 2, 435–461; MR 94j:58003]. For what these curves imply for CC geodesy, see the reviewer's papers [SIAM J. Control Optim. **32** (1994), no. 6, 1605–1620; MR 95g:49006; in *Sub-Riemannian geometry*, 325–339, Progr. Math., 144, Birkhäuser, Basel, 1996; MR 97m:58042] and also [W. Liu and H. J. Sussmann, Mem. Amer. Math. Soc. **118** (1995), no. 564, x+104 pp.; MR 96c:53061; in *Sub-Riemannian geometry*, 341–364, Progr. Math., 144, Birkhäuser, Basel, 1996; MR 98a:58040]. Gromov discusses them briefly (pp. 259–260).

§4.1 contains an exposition on Cartan's prolongation, which Gromov uses as a springboard for conjectures about families of horizontal submanifolds. §4.1 D continues topological work begun by Thom in the case where V is an "affine prolongation"—more precisely the bundle $J^k(M,N)$ of k-jets of maps from M to N, with its canonical distribution. The generic maximal-dimensional horizontal submanifolds of this jet bundle are the k-jets of maps from M to N. Thom began a theory of horizontal chains and homology, which Gromov's begins to extend here. At the beginning of §4.1 we find the basic question: "Can the rank of H be recaptured by 'robust' (e.g. $C^{1-\varepsilon}$-Hölder) metric invariants?" referred to above. In the case $H_2 = TV$ this is simple: the Hausdorff dimension of V is $\operatorname{rank}(H) + 2k_2$ whereas its topological dimension is $\operatorname{rank}(H) + k_2$, so the rank is twice the topological dimension minus the Hausdorff dimension. In the case of depth 3 ($r = 3$) this is an open problem.

Chapter 5 concerns bundles and gauge theory over CC manifolds. It has a preliminary character. Gromov told me it was written in part with the idea of extending some of the miracles of 4-dimensional gauge theory to 5-dimensional contact manifolds.

As is usual with Gromov, the book is unevenly written. Some things are explained with startling clarity, while others I could not begin to penetrate. Various arguments termed "trivial" I could not follow. Sometimes it is hard to see where Gromov is going. These expository shortcomings are far outweighed by sparkling new ideas and points of view, coming at a fast and furious rhythm. One expects that some of the many lines of exploration initiated here will be followed, and may lead to unexpected new results as well as connections between Carnot-Carathéodory geometry and the rest of mathematics.

{For the entire collection see 97f:53002}

Reviewed by *Richard Montgomery*