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NONHOLONOMIC CONTROL AND GAUGE THEORY

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Abstract

We present a dictionary between gauge theory and control theory. This is useful for problems involving the control of the orientations of deformable bodies (robots, gymnasts) by means of shape deformations. In the last section we present some ideas on the stabilization of nonholonomic control systems, where the objective is a given submanifold instead of a single point.

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1 The setting

We will be discussing control laws:

$$\dot{q} = h(q)u \tag{1}$$

linear in the control u. Thus h(q) is a linear operator depending smoothly on the state q. We will be especially interested in cases where q splits locally as

$$q = (x, g) \tag{2}$$

so that the control law has the form

$$\dot{x} = u\,; \, \dot{g} = -A(x, g)u. \tag{3}$$

Moreover g will take values in a Lie group G, usually the rotation group.

Remark 1 *Any control law of the form (1) can be reduced to the form (3) by a smooth feedback transformation* $u \longmapsto \alpha(x)u$*, linear in the controls: provided that*

a) there are fewer controls than states.

b) $h(q)$ *has maximal rank.*

The class of examples we have in mind concerns the attitude control of a deformable body in free fall. The problem of a falling cat righting itself or of a satellite reorienting itself by means of rotors are examples. For these examples x coordinatizes the body's shape and g coordinatizes its orientation relative to a fixed inertial frame. Thus g takes values in the rotation group G of (two or three dimensional) space. $A(x, g)$ is then a matrix which transforms according to

$$A(x, gg_1) = gA(x, g_1) \tag{4}$$

Control law (3) can be rewritten in terms of matrix-valued differential forms:

$$dg + A(x, g)dx = 0 \tag{5}$$

In our class of examples eq.(5) is a rewrite of the statement "angular momentum equals zero".

We are deformable bodies! Imagine that we are in freefall with zero total angular momentum. Our problem is to reorient ourselves, say right-side-up, by changing our shape. Such a problem is faced by gymnasts, falling cats, and unfortunate robots. Our control variables are the deformations dx of our shape. These are affected in turn by exerting torques on joints and extending or retracting limbs and we will henceforth ignore the dynamical problem of implementing the dx's. Our objective is to control g.

A spatial rotation $g \in G$ acts on us by rotation:

$$q \longmapsto gq = q', \tag{6}$$

q being our configuration before rotation and q' our configuration after rotation. The space of all q's forms our configuration space Q. Two shapes are the same if they differ by a rotation. Thus the *shape space* S is the quotient space:

$$S = Q/G$$

Let

$$\pi : Q \to S$$

denote the map which assigns to each configuration q its shape $x = \pi(q)$. We say that G acts freely if whenever $gq = q$ we have that $g = e$ where e denotes the identity of the group. In this case S is a smooth manifold and π gives Q the structure of a principal G-bundle.

Definition 1 *$\pi : Q \to S$ is a principal G-bundle if there is a covering of shape space S by open sets $U \subset S$ together with a family of diffeomorphisms (smooth maps with smooth inverses)*

$$\phi_U : U \times G \to \pi^{-1}(U) \subset Q,$$

$$\phi_U : (x, g) \longmapsto q$$

called "local trivializations" with the property that whenever $q = \phi_U(x, g)$ undergoes the rotation $q \to g_1 q$ then according to the local trivialization ϕ_U we have

$$(x, g) \longmapsto (x, g_1 g)$$

Robot coordinates provide an example of a local trivialization. If there are no constraints on the joints then they actually afford a global trivialization, that is , we can take $U = S$ so that $Q = S \times G$. Consider a robot made of rigid bodies $B_\alpha, \alpha = 1, ..., N$ attached to each other by joints j_i. Abstractly we can think of the robot as a graph with edges B_α and vertices j_i. The shape $x \in S$ is specified by listing the relative joint angles $X_{\alpha\beta} \in G = SO(d)$, where $d = 2$ for planar robots and 3 for spatial ones. If in addition to these joint angles we specify the center of mass c of the entire configuration, or of one of the bodies, and the orientation $g \in G$ of a single one of the bodies **relative to an inertial frame** then we have completely specified the robot's configuration q. If the body is in freefall then $c = c(t)$ is determined beyond our control. Consequently we ignore the center of mass coordinate c, except for the time constraint that it gives us: whatever we decide to do about our orientation we must do before we hit the ground! The global trivialization is then $q \to (x, g) = (X_{\alpha\beta}, g)$.

Remark 2 *Why should we worry about bundles when the configuration space Q is isomorphic to $S \times G$? There are two reasons. The first is that this isomorphism is not canonical. It involved singling out one of the bodies in order to compare its orientation with an inertial frame. Thus we have broken some symmetry (gauge invariance) in the problem and ignoring this symmetry can be detrimental. The other reason is that typically there are numerous and various constraints on the $X_{\alpha\beta}$ according to the incidence relations of the graph, the fact that solid objects cannot pass through each other, and the type of joints which join them. These constraints can lead to nontrivial bundles, in spite of the fact that without the constraints the bundle is always trivial.*

The constraint "angular momentum equals zero" can be written

$$I(x, g)dg + M(x, g)dx = 0 \tag{7}$$

$I(x, g)$ is the locked inertia tensor. This is the moment of inertia tensor of the robot when all of its joints are locked in the shape x and orientation g. $M(x, g)dx$ is the total angular momentum which would result from deforming the joints from x to $x + dx$ without changing g. Our control law is thus of the form of eq. 5 with

$$A(x, g) = I(x, g)^{-1}M(x, g) \tag{8}$$

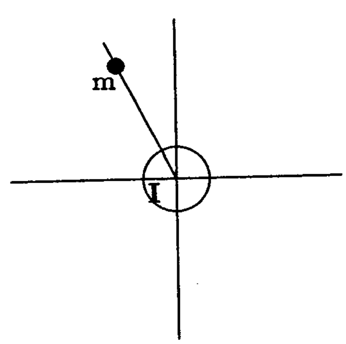
(I is invertible if the action is free.) This last formula is called the 'master formula" or 'master gauge' by Shapere and Wilczek , [25] , [27]. The reader may check that it satisfies the transformation law of eq. (4).

Remark 3 Historical Remark *Formulas 7 , 8 and the fact that they define a connection can be found in Guichardet's paper "On Molecular Dynamics" [14]. They were rediscovered and further exploited by Shapere and Wilczeck [25] [26] [27]. See also Montgomery [20], [21].*

Example 1 Heisenberg's Flywheel Consider a point mass m connected by a massless rod to a flywheel with moment of inertia I. The flywheel is in turn attached to a table by a joint on which the wheel spins freely. The joint is frinctionless so that exerts no torque on the assembly. Then the total angular momentum is zero.

$$\mathbf{I}d\theta + m(xdy - ydy) = 0 \qquad (9)$$

by conservation, assuming it is initially zero. Here θ is the angle of the flywheel relative to the table and (x, y) are the mass's coordinates, both measured with respect to coordinate axes laid out the table. See figure 1, below.



We can exert a torque on the rod to rotate it relative to the wheel and we can slide the mass back and forth on the rod. Thus we have two controls, the torque τ and sliding speed s, and three states (x, y, θ). Eq. (

9) encodes the control law. A linear tranformation of the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{J}(x,y)\begin{pmatrix} \tau \\ s \end{pmatrix},$$

where J is a 2×2 matrix depending algebraically on x and y, will transform the control laws into the form

$$\dot{x} = u$$

$$\dot{y} = v$$

$$\dot{\theta} = -\alpha(xv - yu)$$

where $\alpha = \frac{-I}{m}$. This is the model equation investigated by Brockett in [6], [5].

Suppose we want to optimize the length

$$\int \sqrt{u^2 + v^2} dt$$

of the xy path among all paths with fixed endpoints $q_0 = (x_0, y_0, \theta_0)$ and $q_1 = (x_1, y_1, \theta_1)$. Then the optimal path projects to an arc of a circle in the xy plane. The end point conditions on θ determines the radius of the circle.

$$\theta_0 - \theta_1 = -\alpha \int_\gamma (x dy - y dy) = -2\alpha Area \tag{10}$$

where *Area* is the signed area of the sector formed by the arc together with the radial segments through its endpoints.

To recast the problem in terms of shape variables, put polar coordinates (r, ϕ) on the xy plane. Then eq. (9) reads

$$I d\theta + mr^2 d\phi = 0 \tag{11}$$

Shape space coordinates are r and $\psi = \phi - \theta$. Then:

$$d\theta = \frac{-mr^2}{I + mr^2} d\psi \tag{12}$$

Remark 4 History *We have called this system "The Heisenberg flywheel" because it is isomorphic to a canonical control system on the Heisenberg group H^3 a special three-dimensional nilpotent Lie group. Write:*

$$Q = \frac{\partial}{\partial x} + \alpha x \frac{\partial}{\partial \theta}$$

$$P = \frac{\partial}{\partial y} - \alpha y \frac{\delta}{\delta\theta}$$

$$Z = \frac{\partial}{\partial\theta}$$

Then

$$[Q, P] = \alpha Z \tag{13}$$

with all other Lie brackets zero. (13) is the Heisenberg commutation relative, with $\alpha = \frac{1}{\hbar}$. *The control law is* $\dot{q} = u(t)\, Q(q) \;+\; v(t)\, P(q),\; q \in H^3$.

This model system has been extensively studied by Brockett, [6] , [5].

This optimal control problem goes back to mythological times. Its solution is attributed to Dido. See L.C. Young p. 215, [32]. Actually Dido solved the dual problem: for fixed length maximize the area enclosed by the arc together with the straight line segment joining its endpoints. The solutions to this problem are the same.

2 The Dictionary

We will now present the dictionary. The reader who does not know the theory of principal bundles may need to refer to § 3. The last entry is described in § 4. For additional details see [20], [21]

Control	*Gauge Theory*	*Deformable Body*
control law $\dot{q} = h(q)u$	equations of parallel transport	total angular momentum is zero
	$dg + A(x,g)dx = 0$	
state space	total space Q	configuration space
variables g transverse to contr.	fiber $\pi^{-1}(s)$	all rotations of a given shape s
	structure group G	group of rigid rotations
	Lie algebra of G	space of angular velocities
directly controlled variables x	base space S	shape space
	bundle projection π	to each config. q assigns its shape $x = \pi(q)$
	local trivialization or choice of gauge	robot coordinates are one example
controls u	tangent vectors to base space	shape deformations
$u(t)$ steers from q_0 to q_1	q_1 is the parallel translate of q_0 along $x(t)$ where $\dot{x} = u(t)$	
	the map $q_0 \longmapsto q_1$ is called the holonomy when $x_0 = x_1$	$g_0 \longmapsto g_1$ is the reorientation of the body
Chow's control. criterion	Ambrose-Singer Thm.	What reorientations are possible?
Lie bracket conditions on contr. vector field	curvature conditions	
optimal control for a quadratic cost function	motions of a charged particle in the master Yang-Mills field	most efficient shape deformat. yield. a desired reorient.

3 Connection on Principal Bundles

3.1 Goals and References

The goal of this section is to summarize the theory of a principal bundle with connection and to provide some details of how to get from one column of the dictionary to the other.

For a more detailed treatment of bundles see Steenrod [29]. For a treatment of connections see Bleecker [3] , Chern [8], especially the appendix , or Spivak [28] .

3.2 Bundles

As we have said already, the map $\pi : Q \to S$ which assigns to each configuration its shape is an example of a principal G bundle, provided the G action is free. In definition 1 above we defined principal G bundles and local trivializations.

S is called the base space, Q the total space and the sets $\pi^{-1}(s) \subset Q$ are called the fibers. Thus Q is the union over S of its fibers. Each fiber is diffeomorphic to G, but not in a canonical way. We should think of the fibers as "affine groups", groups G with no preferred identity. This ambiguity in choice of identity is the essence of gauge theory.

A smooth choice of identity $\sigma(s) \in \pi^{-1}(s)$ for each s in some neighborhood is called a 'local section". Thus a local section is a map

$$\sigma : U \subset S \to Q$$

satisfying

$$\pi(\sigma(s)) = s$$

for all x in U. Local sections define local trivializations (definition 1 above) according to the rule

$$\phi_U(x, g) = g\sigma(s)$$

and this defines a one-to-one correspondence between local sections and local trivializations.

In the case of the deformable body the fiber over a shape consists of all configurations having this shape. The ambiguity is that of how to realize a given shape by actually embedding it in space. A local section is thus a choice of reference configuration for each shape in some neighborhood U of shapes.

Remark 5 *In the physics literature the choice of section is often referred to as the choice of gauge. The gauge invariance of a property is then akin to coordinate invariance. A statement or property is called gauge invariant if its validity is independent of the particular choice of local section used to perform a calculation.*

In Kaluza-Klein theories of elementary particle physics S is space-time and G is a space of internal variables attached by π to each point of space-time.

There are topological obstructions to finding an isomorphism $Q \cong S \times G$. The simplest of these are the Chern classes. See [8].

3.3 Connections

The following more algebraic point of view regarding principal bundles is useful in defining connections. Given any point q in Q, let $R_q(g) = qg$ ("right multiplication by q"). Then we have a sequence of maps

$$G \xrightarrow{R_q} Q \xrightarrow{\pi} S \tag{14}$$

of smooth maps between manifolds. We will call this the "bundle sequence". It is an exact sequence of smooth maps in the following sense. The map R_q is one-to-one, the map π is onto, and $\pi^{-1}(s) = image(R_q)$ where $\pi(q) = s$. A local section can be thought of as a local splitting of this exact sequence.

A connection is a family of infinitesimal splittings of the bundle sequence (14). The infinitesimal version of this sequence is obtained by taking differentials of the maps involved:

$$Lie(G) \xrightarrow{\alpha_q} T_qQ \xrightarrow{d\pi_q} T_sS \tag{15}$$

Here we have identified the Lie algebra $Lie(G)$ with T_eG where e is the identity element of the group G, and α_q is the differential of the map R_q : $g \to gq$ at e. α_q is sometimes called the 'infinitesimal action' of G. In the case of deformable body in three-space, $G = SO(3)$, and $Lie(G)$ is the space of infinitesimal rotations. A vector $\omega \in Lie(G)$ represents an instantaneous angular velocity. Thus $Lie(G)$ can be identified with $\mathbb{R}^3$. The infinitesimal action $\alpha_q(\omega)$ is the infinitesimal rotation of the configuration q about the axis ω. In symbols $\alpha_q(\omega)(X) = \omega \times \mathbf{q}(\mathbf{X})$. Here the X's serve to label the points of the deformable body, so that $\mathbf{q}(\mathbf{X}) \in \mathbb{R}^3$ is the inertial position of the body point labelled X when the body is in the configuration q. (Please ignore the difference between bold face and plain q's and X's here. They are all vectors.)

Sequence (15) is an exact sequence of linear maps: α_q is one-to-one, $d\pi_q$ is onto, and $ker(d\pi_q) = image(\alpha_q)$. This can be seen can be obtained by differentiating the condition of exactness of the bundle sequence (14) .

Definition 2 *The image of α_q is called the 'vertical subspace' or simply "vertical space" at q. It represents the space of all rigid deformations of q. It is denoted by V_q.*

We have $V_q = im(\alpha_q) = ker(d\pi_q) = T_q(Gq) = T_q(\pi^{-1}(s))$. Note that α_q provides a canonical isomorphism between $Lie(G)$ and V_q. The union of all the V_q's is called the 'vertical distribution', denoted $V \subset TQ$.

We now give four equivalent definitions of connections.

Definition 3 *A horizontal distribution (sometimes called an Ehresmann connection) is a smoothly varying family*

$$D_q \subset T_qQ$$

of linear subspaces complementary to the vertical distribution and invariant under the G action. Thus

$$T_qQ = V_q \oplus D_q$$

and

$$D_{gq} = gD_q$$

Note that $d\pi_q$ restricted to D_q is a linear isomorphism.

T_qQ is linearly isomorphic to the direct sum $T_sS \oplus Lie(G)$. (This follows from linear algebra and the fact that the infinitesimal bundle sequence, diagram (15) is exact.) However there is, in general, no canonically defined splitting.

Definition 4 *A connection is an equivariant splitting of the infinitesimal bundle sequence (15). Said in more detail, a connection is a family of linear isomorphisms*

$$l_q : T_sS \oplus \ Lie(G) \rightarrow T_qQ$$

depending smoothly on $q \in Q$ *and satisfying the following properties:*

$$d\pi_q(l_q(v,\omega)) = v$$

$$\alpha_q\omega = l_q(0,\omega)$$

(the splitting properties) and

$$l_{gq}(v, g \cdot \omega) = g \ l_q(v,\omega)$$

(the equivariance property).

In these formulas $v \in T_sS$. In the last formula $g \cdot \omega$ denotes the adjoint action of $g \in G$ on ω. (Again, in the deformable body case is the standard action of $SO(3)$ on $\mathbb{R}^3$.) And on the right hand side of this equation $g \ l_q(v,\omega)$ denotes, by abuse of notation, the differential of the map $q \longmapsto gq$ applied to the vector $l_q(v,\omega) \in T_qQ$.

Definition 5 *A horizontal lift is a smoothly varying family of maps*

$$h(q) : T_{\pi(q)} \ S \rightarrow T_qQ$$

such that

$$d\pi_q \circ h(q) = identity \ on \ T_qQ$$

$$h(gq) = gh(q)$$

Definition 6 *A connection one-form is a smoothly varying family*

$$\Gamma_q : T_q Q \to Lie(G)$$

such that

$$\Gamma_q(\alpha_q(\omega)) = \omega$$
$$\Gamma_{gq} = g\Gamma_g g^{-1}$$

(The first g of $g\Gamma_g g^{-1}$ denotes the adjoint action of g on $Lie(G)$. The second g^{-1} represents the action of G on Q.)

These four definitions are equivalent: an object satisfying any one definition canonically defines objects satisfying all others. They are all related by linear algebra. Thus:

$$D_q = im\ h(q) = Ker\ \Gamma_q = l_q(T_{\pi(q)}\ S \oplus O).$$

The reader can work out other relations. For example:

$$l_q = h(q) \oplus \alpha_q$$

By abuse of language, an object satisfying any one of these properties may be called a connection.

In terms of a local trivialization over U the connection one-form Γ must have the form

$$\Gamma(x, g) = (dg + gA(x)dx)g^{-1} \tag{16}$$

Γ is uniquely determined (over U) by the $Lie(G)$ - valued one-form $A(x) \cdot dx$ on U. $A(x) = A(x, e)$ of eq. (3), (5).

Remark 6 *In this last formula we expressed the adjoint action of G on Lie G as $\omega \longmapsto g\omega g^{-1}$ instead of the earlier $\omega \longmapsto g\omega$ Here $\omega = A(x)dx$.*

Warning *Most mathematical texts use right principal bundles as opposed to out left bundles. That is, they write the action of the group as $(q, g) \longmapsto qg$. This leads to several sign differences when formulas are compared to ours.*

There are situations in which Q has a natural connection. One is when it has a Riemannan metric for which the G-action is by isometrices. Then declare

$$D_q = V_q^{\perp}$$

where $V_q = im\, \alpha_q$, and $\perp$ is with respect to the metric on T_qQ. Note that V_q is the tanget space to the G orbit through q, which is also the fiber through q. This is the situation with our class of examples. Take for the Riemannain metric the metric defined by the Kinetic energy of the deformable body.

3.4 Angular Momentum and Riemannian Submersions

There is a nice situtation in which Q has a canonical connection and our deformable body examples fits this situation. Suppose Q is a Riemannian manifold and that G acts on Q by isometries. The vertical space V_q is as before: it is the tangent space to the orbit through q. We define the horizontal distribution to be:

$$\mathcal{D}_q = V_q^{\perp} \tag{17}$$

the orthogonal complement to the vertical space. The invariance of $\mathcal{D}$ under the action of G follows immediately from the fact that G acts by isometries.

If G acts freely then equation 11 defines a connection on the principal bundle $G \to Q \to S = Q/G$. Moreover S inherits a Riemannian metric from Q by declaring that, for each q, The restriction of $d\pi_q$ is an isometry

$$\mathcal{D}_q \to T_sS, \quad s = \pi(q).$$

(Exercise: Show the resulting inner product on T_sS is independent of the choice of $q \in \pi^{-1}(s)$.).

This gives $Q \to S$ then structure of a "Riemannian submersion."

Definition 7 *A submersion $\pi : Q \to S$ is called Riemannian if Q and S have Riemannian metrics such that $d\pi_q$ restricted to $\mathcal{D}_q = ker(d\pi_q)^{\perp}$ is an isometry for each $q \in Q$.*

In the case of a deformable body there is a canonical Riemannian metric defined by the kinetic energy:

$$< \delta q_1, \delta q_2 >= \int_{X \in B} < \delta q_1(X), \delta q_2(X) > dm(X) \tag{18}$$

where B denotes a reference body, dm is the mass distribution and $X \in B \longmapsto \delta q_i(X) \in \mathbb{R}^3$, i = 1,2, are two deformations of the body, i.e. $\delta q_i \in T_q Q$.

If $\delta q_2 \in V_q \subset T_q Q$ is a vertical vector then there is an $\omega \in \mathbb{R}^3$ such that

$$\delta \mathbf{q}_2(x) = \omega \times \mathbf{q}(X))$$

And so

$$< \delta q_1, \delta q_2 >=< \omega, \mathbf{M}(q, \delta q) >$$

where

$$\mathbf{M}(q, \delta q_1) = \int_B (\mathbf{q}(X) \times \delta \mathbf{q_1}(X)) dm(X)$$

is the standard expression for the total angular momentum associated to the deformation $\delta \mathbf{q_1}$ of the configuaration q. It follows that $\delta \mathbf{q_1}$ is horizontal iff $\mathbf{M}(q, \delta \mathbf{q_1}) = 0$. Thus

$$D_q = \{\delta q \in T_q Q : \mathbf{M}(q, \delta q) = 0\}.$$

This is the basic fact which makes the language of connections on principal bundles useful for the control of deformable bodies.

If the horizontal distribution $\mathcal{D}$ is defined by the vanishing of the angular momentum $\mathbf{M}$ then the connection one-form Γ has the same kernel as $\mathbf{M}$. Consequently we must have :

$$\Gamma(q) = R(q)\mathbf{M}(q, \ )$$

for some invertible transformation $R(q) : Lie(G) \rightarrow Lie(G)$. [Note: that Γ and $\mathbf{M}$ are both one-forms on Q with values in $Lie(G)$ and they obey the same transformation law with respect to the G action.] To find $R(q)$ we use the normalization condition $\Gamma(q)(\alpha_q(\omega)) = \omega$ for the connection one-form. Now

$$\mathbf{M}(q, v) = \alpha_q^t(v)$$

where

$$\alpha_q^t : T_qQ \to Lie(G) = \mathbb{R}^3$$

is the transpose of the map

$$\alpha_q : Lie(G) \to T_qQ$$

(α_q is the infinitesimal G-action). The normalization condition becomes $R(q)\alpha_q^t\alpha_q =$ identity. Therefore $R(q) = (\alpha_q^t\alpha_q)^{-1}$. Now $\alpha_q^t\alpha_q = I(q)$ is the locked inertia tensor. This yields the "master formula" equation (7), (8).

3.5 Parallel Transport: the Control Law

Definition 8 *A vector or vector field is said to be horizontal if it lies in D. A path is said to be horizontal if its derivatives all lie in D.*

Our control law is that curves be horizontal.

With h denoting the horizontal lift operator, this control law is our original equation, (1). Now it is called the equation of parallel transport. It can be rewritten in the following ways

$$q^*\Gamma = 0$$

(cf. with eq. (5)) or

$$\dot{q} \in D_q(t)$$

The second of these has exactly the form of eq. (3) when written out in a trivialization. (Cf. eq. (16).)

Any of these equations is called the equation of parallel transport. A solution $q(t) = (x(t), g(t))$ is called "the parallel transport of $q^{(0)}$ along $x(t)$". Solving the equations of parallel transport for different initial conditions $q(0) \in Q$ defines a map

$$\mathbb{P}_\gamma : Q_0 \to Q_1$$

where

$$Q_0 = \pi^{-1}(x(0)), Q_1 = \pi^{-1}(x(1)),$$

and

$$\mathbb{P}_\gamma q(0) = q(T).$$

Here γ denotes the curve $x[0,T]$ in shape space. $\mathbb{P}_\gamma$ is called the parallel transport operator along the curve γ. It is a crucial and easily proved fact that this operator is independent of how the curve γ is parameterized.

In the case of the deformable body $\mathbb{P}_\gamma$ describes by how much the body has rotated due to the sequence of shape changes $x(t)$. This interpretation is most meaningful when

$$x(0) = x(T)$$

so that the final and initial shapes are the same. Then the configurations $q_0 = q(0)$ and $q_1 = q(T)$ differ by a rigid rotation $g_1 = g(T)$:

$$\mathbb{P}_\gamma \mathbf{q}(\mathbf{0}) = \mathbf{q}(\mathbf{T}) = \mathbf{g}(\mathbf{T})\mathbf{q}(\mathbf{0})$$

In this case the parallel transport map is called the holonomy. It describes the net reorientation of the body.

The parallel transport operators satisfy

$$\mathbb{P}_\gamma(\mathbf{gq}) = \mathbf{g}\mathbb{P}\gamma(\mathbf{q})$$

and

$$\mathbb{P}_{\gamma_2\gamma_1} = \mathbb{P}\gamma_2 \circ \mathbb{P}\gamma_1$$

In terms of the deformable body, this first identity says that if we rigidly rotate our body then perform a sequence of shape changes then the resulting reorientation is the same as if we had first performed the sequence of shape changes and then rotated the body. In the second identity γ_1 and γ_2 are two paths such that the end point of γ_1 is the initial point of γ_2. Then $\gamma_1 \cdot \gamma_2$ stands for the single path obtained by joining the two paths together at this common endpoint.

The parallel transport equations (3, 1) are solved in a local trivialization by letting $M(t)$ be the fundamental solution to equation (3). Thus $M(0)$ =identity. Then g(t)=g(0)M(t) so that

$$\mathbb{P}(\mathbf{x}(\mathbf{0}), \mathbf{g}(\mathbf{0})) = (\mathbf{x}(\mathbf{T}), \mathbf{g}(\mathbf{0})\mathbf{M}(\mathbf{T}))$$

in the given trivialization.

Remark 7 *In the physics literature one finds the notation*

$$M(t) = P\ \exp\{- \int_0^t A(x)dx\}$$

for the fundamental solution, and hence for the parallel transport generator.

"P exp" stands for "path-ordered exponential" it has the following meaning. Consider partitions $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq t$ *of [0,t] for which the mesh size goes to 0 as* $n \to \infty$ *as in the definition of the Riemann integral. Set*

$$\alpha(s) = A(x(s)) \cdot \frac{dx}{dt}(s).$$

Then

$$Pexp - \int_0^t Adx = \lim_{n\to\infty} \exp(-\alpha(t_1))\exp(-\alpha(t_2))...(\exp(-\alpha(t_n))$$

3.6 Controllability and Curvature

Set

$$X_i(q) = h(q) \cdot e_i$$

where $\{e_i\}$ is a basis for the set of controls (a smooth local frame field for S) and where h is the horizontal lift (control law eq. (1)). For example we might take $e_i = \frac{\partial}{\partial x^i}$, the coordinate vector fields, when $(x^1, ..., x^n)$ are (local) coordinates on the shape space S. The X_i are smooth horizontal vector fields on Q. They form a frame (pointwise basis) for the distribution D = image h.

Recall the following consequence of Chow's theorem:

Theorem 1 *Suppose that* Q *is connected. If the* X_i *together with all of their iterated Lie brackets* $[X_i, X_j], [[X_i, [X_j, X_k]]...$ *eventually span the tangent space of* Q *at every point. Then our system 1 provides accesibility: given any two points* $q_0, q_1 \in Q$ *there is a control law* $u(t)$ *which steers from* q_0 *to* q_1

Remark 8 *This theorem is, of course, valid for a general distribution, i.e. one not necessarily coming from a connection on a bundle. For treatments of the theorem, see Chow [9], Rashevski [23], or Sussmann, §3.2 [31].*

One can algebracize the computations by thinking of the Lie bracket of horizontal vector fields at q as a map

$$[\cdot,\cdot]_q : D_q \times D_q \to T_qQ/D_q.$$

To do this take $v, w \in D_q$ and extend them to smooth horizontal vector fields V, W defined in a neighborhood of q. (Recall that horizontal means takes value in D.) Then set

$$[v,w]_q = [V,W](q) mod D_q$$

. To see that this is well-defined observe that

$$[fV, gW]_q = f(q)g(q)[V,W](q) mod D_q$$

for any smooth functions f and g, i.e. the operation is "tensorial": it really only depends on $v = V(q)$ and $w = W(q)$ and not on their horizontal extensions V, W.

Similar operations can be used to make tensorial sense out of higher brackets. See Gershkovich and Vershik [12].

When D is the horizontal space of a connection on a principal bundle then we have the splitting

$$D_q \oplus Lie(G) \cong T_qQ$$

so that we can and do identify

$$T_qQ/D_q = Lie(G).$$

Since $D_q = ker\Gamma_q$ the identification is provided by the connection one-form. Moreover we can use the horizontal lift operator h_q to identify $T_{\pi(q)}S$ with D_q.

Definition 9 *The curvature form F at q is the negative of the Lie bracket $-[\cdot,\cdot]_q$ at q after applying the above identifications. Specifically:*

$$F(q)(v,w) = -\Gamma_q([h_q w, h_q w]_q)$$

Here $v, w \in T_{\pi(q)}S$. Thus $F(q)$ is a skew-symmetric bilinear form (two-form) on $T_{\pi(s)}S$ with values in the vector space $Lie(G)$.

The transformation law for the connection one-form Γ implies that

$$F(gq) = gF(q)g^{-1}$$

This transformation law is equivalent to saying that F is a two-form on S with values in the "adjoint bundle" which is a certain vector bundle over S with typical fiber $Lie(G)$. Alternatively, by using $h_q \circ d\pi_q$ instead of h_q in the definition, we can think of F as a two-form on Q with values in the vector space $Lie(G)$. As such one calculates:

$$F = d\Gamma - [\Gamma, \Gamma]$$

or in a local trivialization

$$F(x, g) = g(dA - [A, A])$$

Here

$$[\Gamma, \Gamma](v, w) = [\Gamma(v), \Gamma(w)]$$

and

$$= \Gamma(v) \times \Gamma(w) \text{ for } G = SO(3)$$

with the last Lie brackets being those in $Lie(G)$.

To prove these formulas use the Cartan's formula

$$(d\Gamma)(X, Y) = (d(\Gamma(Y)))(X) - (d(\Gamma(X)))(Y) - \Gamma([X, Y])$$

for the exterior derivative d and apply it to horizontal vector fields.

Remark 9 *If G is Abelian, for example $G = SO(2)$, then $gF(q)g^{-1} = F(q)$ so that the curvature is an old-fashioned two-form on S. It can be defined by $\pi^* F = d\Gamma$.*

The curvature form is a covariant derivative of the connection form. We can directly relate higher covariant derivatives of the curvature form

$(D_X F)(Y,Z), (D_X D_Z F)(Y,W)$ etc., to the higher intrinsic Lie brackets $[h_q X, [h_q Y(q), h_q Z]]_q, [h_q X, [h_q Z, [h_q Y, h_q W]]]_q$

As a result of these relations we have the following consequence of Chow's theorem [9](See Hermann [16] for more details. Also, somewhere in Hermann's 24 volume set.)

Theorem 2 *[Ambrose-Singer Theorem] Suppose that Q is connected. Suppose that for some $q \in Q$ the image of the curvature Γ_q together with all of its covariant derivatives span Lie(G). Then any two points q_0, q_1 of Q can be joined by a horizontal path.*

Remark 10 *The original The Ambrose-Singer theorem is more general than this. We have given a computationally useful version just as we did with the original Chow theorem.*

Example 2 *(again)* Here $A = \alpha(xdy - ydx)$ The Lie algebra is one dimensional and so Abelian: there is no $[\Gamma, \Gamma]$ term. Thus

$$F = dA = \alpha dx \wedge dy$$

As long as $\alpha \neq 0$ any two points can be joined by a horizontal path.

4 Optimal Control

Consider a pointwise cost of the form

$$c = \frac{1}{2} R(x)(u,u)$$

and corresponding cost functional

$$C[x,u] = \int_0^T c(x(t), u(t))dt.$$

$R(x)(\cdot,\cdot)$ is an inner product (positive definite quadratic form) on controls which depends smoothly on $x \in S$. In other words, it is a a Riemannian metric on the shape space S. The cost C of a control strategy is thus the corresponding integrated kinetic energy. Or, what is effectively the same, the cost is the length of the corresponding projected path $x(t) = \pi(q(t))$ in S. The problem is to minimize C among all controls steering from q_0 to q_1 in time T.

Equivalently, we could fix shapes $x_0 = \pi(q_0), x_1 = \pi(q_1)$ and a parallel transport operator $\mathbb{P} \in \mathbf{Hom}(\mathbf{Q_0}, \mathbf{Q_1})$ where $Q_i = \pi^{-1}(x_i)$. These two types of boundary conditions are related by $\mathbb{P}(\mathbf{q_0}) = \mathbf{q_1}$. Thus I call the problem the "isoparallel problem". In case $x_0 = x_1$ it is the isoholonomic problem. Example 1 is the isoarea problem which is dual to the famous isoperimetric problem.

We may think of the cost C as the 'efficiency' of a given control strategy. Thus our optimal control problem is to find the most efficient way to deform a deformable body ao as to achieve a desired reorientation. This problem was first formulated by Shapere and Wilczek [25] [26] in connection with the question of how certain microorganisms swim. They solved the corresponding linearized problem, that is, the case of infinitesimal shape deformations.

¿From a geometric point of view the natural cost to use is the length of the path in shape space, or what is effectively the same, its integrated kinetic energy. These are to be calculated with respect to the metric defined in §3.4. The optimal control problem then becomes the problem of finding that horizontal path which connects q_0 to q_1 in a time T and which minimizes the integrated kinetic energy over this time interval.

The basic facts regarding the extremals to this problem are

Theorem 3 *[20] The normal extremals for the above optimal control problem obey the same differential equations as those of a particle under the influence of the gauge potential A (cf. eq. (16)) travelling on the Riemannian manifold (S, R). In case the group G is $SO(2)$ (planar robot) these are the standard Larentz equations which govern the motion of a charged particle in the magnetic field $F = dA$.*

Theorem 4 *[22], [20] The abnormal extremals are those horizontal curves*

$q(t)$ such that there exists a nonzero element μ in the dual of Lie(G) such that $\mu(F(\dot{q},\cdot)) = 0$.

In case dim$(Q) = 3$, dim$(G) = 1$ (for example $G = SO(2)$) so that dim$(S) = 2$ this condition implies that the projection $x(t) = \pi(q(t))$ lies on the zero level set of the magnetic field B where $F = BdS$ and dS denotes the area form on S. If this level set is generic ($dB \neq 0$ when $B = 0$) then every sufficiently short subarc of $q(t)$ is the unique cost minimizing path between its endpoints. Moreover in this case the abnormal extremal is stable under perturbations of the cost and the control law (eq. 1) .

Remark 11 (Explanation) *"Abnormal" refers to the fact that we can take $\lambda_0 = 0$ when we compute the extremals of $\lambda_0 C + \lambda G$. Here C is the cost function and G represents the horizontal constraints on the paths. See Bliss [4], Hermann [16], and especially Morse and Myers [19] for more information on normal versus abnormal curves.*

The optimal control Hamiltonian H is the Hamiltonian defining the differential equation for theorem 3.It is the Hamiltonian furnished us by Pontrjagin's principle in the normal case. To write it down, choose a local frame of vector fields X_μ for S and calculate the matrix of inner products

$$g_{\mu v}(x) = R(x)(X_\mu(x), X_v(x))$$

let $g^{\mu v}$ be the inverse matrix. Then

$$H = \frac{1}{2} g^{\mu v}(hX_\mu)(hX_v)$$

where, as before, h denotes horizontal lift. Also, the hX_μ, being vector fields on TQ, are fiber-linear functions on T*Q. Thus H is a fiber-quadratic function on T*Q.

5 Two Examples

Example 3 *Example 1, revisited*

$$X_1 = \frac{\partial}{\partial x} = (1, 0, 0)$$

$$X_2 = \frac{\partial}{\partial y}$$

$$hX_1 = \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial \theta}$$

In coordinates these are the fiber linear functions

$$hX_1 = p_x + \alpha y p_\theta$$

$$hX_2 = p_y - \alpha x p_\theta$$

(Just replace $\frac{\partial}{\partial x}$ with p_x, etc.)

$$H = \frac{1}{2}\{(p_x + \alpha y p_\theta)^2 + (p_y - \alpha x p_\theta)^2\}$$

This is the Hamiltonian for a particle of charge p_θ is a constant magnetic field of strength α. The solutions are circles as any textbook on electromagnetism will say.

Example 4 The Falling Cat of Kane and Scher

Kane and Scher modeled the maneuver by which the falling cat, dropped from upside down with zero angular momentum, rights itself. Their model consists of two identical axially symmetric rigid bodies joined along their symmetry axes by a special kind of joint ("no-twist").

If the joint were instead a ball-and-socket joint (three degrees of freedom) and if we ignored collisions of the bodies then the configuration space would be

$$Q_{b-s} = So(3)_f \times SO(3)_b.$$

The subscripts "f" and "b" stand for "front" and "back". The subscript "b-s" is for ball-and-socket.

Kane and Scher make the following beautiful choice of coordinates. Let 1_f and 1_b denote the axis of symmetry, the backbone, of the front and back body halves. Let ψ denote the angle between 1_b and 1_f. As long as $\psi \neq 0, \pi$. 1_f and 1_b define a plane $\wp$ the plane which contains them and the joint. The angles θ_f and θ_b will denote the angles made by the respective feet axes

2_f and 2_b and this plane. The orientations are chosen so that θ_f increases as the front rotates about the 1_f axes, orientated toward the head, according to the right hand rule. And θ_b increases as the back half rotates about the 1_b axis, oriented to point toward the tail. Then $(\psi, \theta_f, \theta_b)$ coordinatize the shape space S_{b-s} where the subscript stands for ball-and-socket joint. To define coordinates on Q_{b-s} we need a local section $s : S_{b-s} \to Q_{b-s}$. This induces coordinates as in §3.1 :

$$(\psi, \theta_f, \theta_b, g) \longmapsto g\, s\,(\psi, \theta_f, \theta_b).$$

We choose the section s (choice of gauge) by insisting that the plane $\wp$ is the yz plane, that 1_f makes an angle $\psi/2$ with the positive z-axis, that 1_b makes the angle $-\psi/2$ with the same, that when the coordinates $\theta_f = \theta_b = 0$ then the cat's legs are lying in the plane, and that if in addition $\psi = \pi$ so that the cat is fully extended (backbone straight) then the feet are pointed straigt up (cat upside-down). The negative z axis is the direction of gravity.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the orthonormal frame in the inertial frame associated to these axes. e (no bold-face) will denote the identity matrix corresponding to choice of section s. Set

$$c = cos(\psi/2) \qquad s = sin(\psi/2)$$

Let I_3 denote the moment of inertia of either body for resisting spinning about its symmetry (1_f or 1_b) axis. Then the total angular momentum M generated by the deformation $(\psi, \theta_f, \theta_b, e) \to (\psi + d\psi, \theta_f + d\theta_f, \theta_b + d\theta_b, e)$ is

$$M = I_3\{[c\mathbf{e}_3 + s\mathbf{e}_2]d\theta_f + [c\mathbf{e}_3 - s\mathbf{e}_2]d\theta_b\}.$$

The moment of inertia tensor of the configuration $s(\psi, \theta_f, \theta_b)$ with respect to the inertial frame is the diagonal matrix

$$I(\psi) = 2I_1 diag[c^2 + (1+\beta)s^2, c^2 + \alpha s^2, \alpha c^2 + (1 + \beta s^2)]$$

where $I_1 = I_2$ is the moment of inertia of either body half with respect to any axis orthogonal to the symmetry axis, and

$$\alpha = \frac{I_3}{I_1}$$

is the ratio of moments of inertia and

$$\beta = \frac{ml^2}{I_1}$$

where l is the distance of the center of mass of either body half from the joint and m is the mass of either body half.

Plugging these results into the master formula (8) for the connection one-form where x stands for $(\psi, \theta_f, \theta_b)$ we obtain

$$A(x) = \frac{\alpha}{2}\frac{c(d\theta_f + d\theta_b)}{\alpha c^2 + (1+\beta)s^2}\mathbf{e}_3 + \frac{\alpha}{2}\frac{s(d\theta_f - d\theta_b)}{c^2 + \alpha s^2}\mathbf{e}_2$$

Recall that the vector coefficients $\mathbf{e}_i$ refer to infinitesimal rotations about these axes.

The no-twist condition of Kane and Scher is

$$d\theta_f = -d\theta_b.$$

We can think of this as saying that the cat is not allowed to break its own back. Integrating we obtain

$$\theta_f = -\theta_b$$

and so we set

$$\theta := \theta_f.$$

We would like to thank Mark Enos at this point for pointing out that the no-twist condition is **not** the same as connecting the two halves by a U -joint (sometimes called a Hooke's joint)!

The full connection-one form $dg + gA(x)$ for the no-twist joint is then

$$\{d\chi + \Phi(\psi)d\theta\}\mathbf{e}_2$$

where

$$\Phi(\psi) = \alpha\frac{s}{c^2 + \alpha s^2}$$

and where we have written $g = exp(\chi\mathbf{e}_2)$= rotation about the y-axis by an amount χ.

We have just proved the remarkable fact, observed by Kane and Scher, that for no-twist deformations the connection one-form is Abelian. Consequently the re-orientation $\Delta\chi$ of the model cat due to a given shape deformation can be calculated by a single quadrature. In bundle-theoretic language, what has happened is that by restricting the joint to be no-twist we reduce the structure group of the bundle from $SO(3)$ to the (disconnected) group $O(2)$.

The map $\pi : Q \to S$ in the no-twist case is the Hopf fibration $SO(3) \to \mathbb{R}\mathbf{P}^2$, where $\mathbb{R}\mathbf{P}^2$ denotes the real projective plane. This is another surprise: the shape space for a no-twist joint is the real projective plane! This fact can be seen algebraically by using Euler angle (i.e. exponential) coordinates on $SO(3)$, the ball-and-socket shape space, to express the no-twist joint constraint. Then the no-twist shape space becomes identified with $exp(\mathbb{R}^2) \subset \mathbf{SO(3)}$ where $\exp : \mathbb{R}^3 = \mathbf{Lie(SO(3))} \to \mathbf{SO(3)}$ is the usual exponential map and $\mathbb{R}^2$ is the xz plane. If we do not let the body halves pass through each other then we destroy all topology: the shape space becomes diffeomorphic to a disc. In fact, if we imagine the body halves to be infinitely thin rods then shapes we must delete are those with $\psi = 0$. This is a circle of shapes (vary θ) and corresponds to the line at infinity in the classical conception of the projective plane. Thus the shape space in this case is a classical affine plane.

The normal extremals for the optimal control problem are the trajectories of charged particles travelling on the real projective plane with metric defined by the pointwise cost function. For more details concerning this problem consult a forthcoming paper or write the author.

The abnormal extremals are the horizontal lifts of certain lines of lattitude $\psi = constant$ where the constants are the solutions to the equation

$$\frac{d\Phi}{d\psi} = 0.$$

6 Restrictions and Uses of the Dictionary

6.1 Restrictions

The control law must be linear in the control in order to be able to use the dictionary. A partial extension exists for laws which are affine in control.

We require the control law $u \longmapsto h(q)u$ to be maximal rank for all q. In some cases this could be a significant restriction. We require the number n of state variables to be greater than the number m of controls. (The case $m \geq n$ is trivial from a theoretical point of view.) Any control system satisfying these propertites can always be put in local bundle form of equation (3).

The dictionary is most useful for systems with some symmetry, namely that of equation (4). It is even more useful if the number $n - m$ (the dimension of G) is much less than the number m of controls.

6.2 Utility

Some computations become extremely easy. For example we have shown that using ideas from gauge theory it can be very easy to check controllability. The dictionary allows easy identification of important special curves and submanifolds. For example, singular arcs lie on the surfaces {curvature $= 0$ }. The dictionary can be used as an aid to intuition. For example, solutions to the naturally formulated optimal control problems (quadratic cost) are trajectories in a magnetic field defined by the curvature.

By using the dictionary, or at least its geometric philosophy, we can find the correct formulations of some open problems in nonholonomic path planning, for example the problem of feedback stabilization. See the next section.

7 Feedback Stabilization

Suppose the point $q_0 \in Q$ is the desired goal. For example, it might represent the cat's desired configuration: feet pointing down back slightly arched. In feedback stabilization we try to design a feed back control law

$$u = u(q)$$

which steers all points in some neighborhood of q_0 to q_0. Such a feedback law is called a feedback stabilization strategy. We now cite the following basic theorem of Brockett. [6].

Theorem 5 *[Brockett] Let $\dot{q} = h(q,u)$ be the control law. If the map $(x,u) \longmapsto h(x,u)$ does not map a neighborhood of $\{q_0\}\times$ controls onto a neighborhood of 0, then there is no continous feedback stabilization strategy.*

In our situation h can always be put in the form:

$$(x, g; u) \longmapsto (u, -A(x,g)u) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$$

Points of the form $(0,\xi), \xi \neq 0$ are never in the image of this map. Thus our systems never admit continuous feedback stabilization laws.

However, if we give up the idea of stabilizing to a point and instead pick a desired subvariety to stabilize onto we can achieve victory.

Example 5 Rewrite the Heisenberg flywheel in polar coordinates: (r, ϕ) for the xy-plane. Let us set $z = \theta$. We will think of (r, ϕ, z) as cylindrical coordinates on three-space for the purposes of visualization. The system becomes

$$\dot{r} = u_1$$
$$\dot{\phi} = u_2$$
$$\dot{z} = -\frac{1}{2}r^2 u_2$$

Our goal is to stabilize onto a circle of radius r_1 at a height z_1. Set

$$u_1 = -c(r - r_1)$$

$$u_2 = z - z_1$$

One easily calculates that the distance squared

$$V = (r - r_1)^2 + (z - z_1)^2$$

of a point from our objective circle is a Liapanov function. That is, $\frac{dV}{dt} \leq 0$ everwhere and $\frac{dV}{dt}(q) = 0$ only for points on the unit cylinder. Thus we have stabilized the system onto our circle by means of a feedback stabilization law. Note that in order to do this we gave up all control of our position on that circle.

In terms of our mass fly-wheel model we can feedback stabilize the distance r of the mass from the fly wheel and the inertial angle of the flywheel but not the angle of the rod. By a alternative strategy we could have stabilized instead the rod angle or the difference of the two angles ψ.

Example 6 Brockett has proposed the followoing generalization of Heisenberg's flywheel:

$$\dot{x} = u$$

$$\dot{\xi} = x \wedge u$$

Here $x, u \in \mathbb{R}^n$ and $\xi \in Lie(SO(n)) = \Lambda^2\mathbb{R}^n$. The feedback law

$$u = -\xi x$$

stabilizes the sytem onto the subvariety

$$\{(x, \xi) : \xi x = 0\}.$$

A general framework is a follows. Suppose that our objective is to feedback stabilize control system (1) onto the submanifold $N \subset Q$. We say that N is transverse to the horizontal distribution D if

$$T_qN + D_q = T_qQ \text{ for all} q \in N.$$

Let us suppose that N is defined by some equation: $N = p^{-1}(y_0)$ where

$$Q \xrightarrow{p} Y$$

is a smooth map and y_0 is a regular value of Y, that is, $dp(T_qQ) = T_{p(q)}Y$ whenever $p(q) = y_0$. If N is a smooth manifold with trivial normal bundle then it is always possible to write N in this way. We think of Y as the objectives. In our first stabilization example $p(r, \phi, z) = (r, z)$.

The transversality condition on D and N translates to

$$dp_q D_q = T_{p(q)}Y$$

for $q \in N$. Now we suppose that $dim(Y) = rank(D)$ which is the same as

$$dim(N) + \text{number of controls} = dim(Q).$$

Then $dp_q \circ h(q)$ is a linear isomorphism of the controls u onto the objectives T_yY. The control law (1) induces the law

$$\dot{y} = \frac{\partial p}{\partial q}(q)h(q)u$$

for the objectives. ($\frac{\partial p}{\partial q}$ and dp_q are different symbols for the same thing.) Choose coordinates so that $y_0 = 0$. Then the feedback law

$$u(q) = -\Big[\frac{\partial p}{\partial q}(q) \circ h(q)\Big]^{-1} \cdot p(q) \tag{19}$$

yields the differential equation $\dot{y} = -y$. We have proved:

Theorem 6 *Consider control law (1). Suppose $N \subset Q$ is a submanifold which is transverse to the control distribution and whose normal bundle is trivial. Then feedback law (19) (locally) stabilizes the system onto N.*

As a trivial illustration of the theorem take $p : Q \to Y$ to be the bundle projection $\pi : Q \to S$ to be the bundle projection. One easily checks the hypothesis. The induced control system on S is just $\dot{x} = u$ which is obviously feedback stabilizable.

Remark 12 *Independent of this work, Bloch and Rehyhanoglu and McClamroch came up with essentially the same idea for stabilization onto submanifolds. See their preprint [2].*

As a final remark we would like to advertise a very recent result of Coron which is much deeper than the just proved theorem.

Theorem 7 *[Coron's Theorem] Suppose that the distribution D is bracket generating and that $Q = \mathbb{R}^n$. Then there exists a time dependent feedback control*

$$u = u(x, t)$$

which is periodic in time:

$$u(x, t) = u(x, t + \tau)$$

and under which the origin of Q becomes globally asymptotically stable. The period τ can be any fixed positive number.

Remark 13 *The proof of Brockett's theorem is topological. The index of the vector field $x \longmapsto h(x)u(x)$ is an obstruction to stabilization. It seems that the essence of Coron's idea is that by suspending the control system to one on $Q \times S^1$ the obstructions vanish. Coron's theorem is of course also true if Q is a general manifold and global stability is replaced by local stability.*

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