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CONVEXITY OF THE FIGURE EIGHT SOLUTION TO THE THREE-BODY PROBLEM

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The Newtonian three-body problem with equal masses has a remarkable solution where the bodies chase each other around a planar curve having the qualitative shape and symmetries of a figure eight. Here we prove that each lobe of this curve is convex.

1. Introduction

The figure eight is a recently discovered periodic solution to the Newtonian three-body problem in which three equal masses traverse a single closed planar curve in the form of an 8 (Figure 1). See [Moore 1993; Chenciner and Montgomery 2000]. The curve has one self-intersection, the origin, which divides it into two symmetric lobes. In [Chenciner and Montgomery 2000] it was proved that each lobe is star-shaped. Here we prove the lobes are convex. (A computer proof based on interval arithmetic appears in [Kapela and Zgliczyński 2003].)

Theorem 1. *Each lobe of the eight solution is a convex curve.*

In the final section we describe how the theorem generalizes to prove the convexity of eights for many three-body potentials besides Newton's.

2. Preliminaries

We present a number of properties of the eight established in [Chenciner and Montgomery 2000] and three assertions relating mechanics and plane geometry. The convexity proof relies on these properties and assertions.

Center of Mass. Write $q_1(t), q_2(t), q_3(t)$ for the location of the three masses in the plane at time t. At each time t we have $q_1(t)+q_2(t)+q_3(t)=0$.

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Symmetry. Write $R_y(x, y) = (-x, y)$ for the reflection about the y axis. Then the eight solution enjoys the following symmetries:

$$\begin{aligned}\big(q_1(t), q_2(t), q_3(t)\big) &= \big(R_y\big(q_3(t - \tfrac{1}{6}T)\big), R_y\big(q_1(t - \tfrac{1}{6}T)\big), R_y\big(q_2(t - \tfrac{1}{6}T)\big)\big),\\ \big(q_1(t), q_2(t), q_3(t)\big) &= \big(-q_1(-t), -q_3(-t), -q_2(-t)\big).\end{aligned}$$

The right-hand side of these equations defines transformations s and σ on the space of all T-periodic loops. These transformations generate an action of the dihedral group

$$D_6 = \{s, \sigma \mid s^6 = 1,\ \sigma^2 = 1,\ s\sigma = \sigma s^{-1}\},$$

the symmetry group of a regular hexagon, which is consequently a symmetry group of the eight.

Invariance under $s^2 \in D_6$ implies that $\big(s^2(q_1, q_2, q_2)\big)(t) = \big(q_1(t), q_2(t), q_3(t)\big)$. Setting $q = q_1$ this last equation reads

$$q_1(t) = q(t), \quad q_2(t) = q\big(t + \tfrac{1}{3}T\big), \quad q_3(t) = q\big(t + \tfrac{2}{3}T\big). \tag{1}$$

A *choreography* is a three-body solution satisfying (1). The curve $q(t)$ is the curve of the eight whose lobes are the subject of Theorem 1.

The D_6-invariance of the figure eight implies that it is completely determined by the three arcs $q_1\big([-\frac{1}{12}T, 0]\big)$, $q_2\big([-\frac{1}{12}T, 0]\big)$, $q_3\big([-\frac{1}{12}T, 0]\big)$ swept out by the three masses over the time interval $[-\frac{1}{12}T, 0]$. *To prove Theorem 1 it is enough to prove that the curvatures of these three arcs are never zero* (with the exception of the point $q_1(0)$, the self-intersection point of the eight, which is taken to be the origin).

A configuration (q_1, q_2, q_3) satisfying $q_1 + q_2 + q_3 = 0$ is called an *Euler configuration* if one of the q_i vanishes. Then necessarily the other two masses q_j, q_k are of the form $\zeta, -\zeta$, so that the entire configuration (q_1, q_2, q_3) is collinear with mass i at the origin located at the midpoint of the segment defined by the other two masses j and k. Upon translating time if necessary, and relabeling the masses, we can insist that at time 0 the configuration is an Euler configuration with mass 1 at the origin and 3 in the first quadrant, as indicated in Figure 1. At the initial time $t = -\frac{1}{12}T$ the three masses form an isosceles triangle, with mass 2 at the vertex and lying on the negative x-axis.

The eight minimizes the usual action of mechanics (integral of the kinetic minus potential energy) among all T-periodic loops enjoying D_6 symmetry. Equivalently [Chenciner and Montgomery 2000] the path $\big(q_1(t), q_2(t), q_3(t)\big)$ of the eight over the fundamental time interval $\big[-\frac{1}{12}T, 0\big]$ minimizes the action among all paths starting at time $-\frac{1}{12}T$ in an isosceles configuration with 2 being the vertex and ending at time 0 in an Euler configuration with 1 being the origin. An important consequence of minimization, proved in [Chenciner and Montgomery 2000,

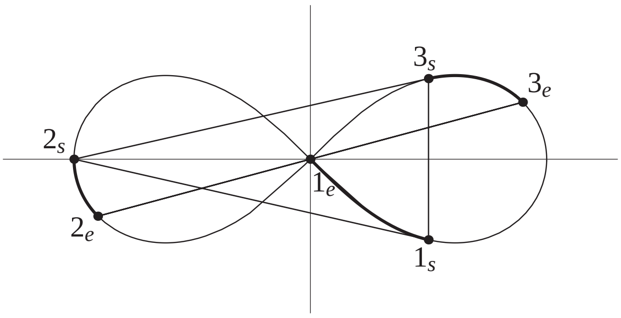

Figure 1. The eight. The labels 1_s and 1_e represent the location of mass 1 at $t = -\frac{1}{12}T$ and $t = 0$, and likewise for 2 and 3.

pp. 896–897], is that there are no times in the fundamental domain besides the endpoints at which the configuration is either collinear or isosceles. It follows that, for all $t \in \left(-\frac{1}{12}T, 0\right)$,

$$r_{13} < r_{12} < r_{23} \tag{2}$$

and

$$q_1 \wedge q_2 = q_2 \wedge q_3 = q_3 \wedge q_1 < 0, \tag{3}$$

where $r_{ij} = |q_i - q_j|$ is the distance between masses i and j and we write

$$(x, y) \wedge (u, v) = xv - yu$$

for planar vectors (x, y) and (u, v). We call equation (2) the *distance ordering inequality*.

Initial and final velocities. At the Euler time, $t = 0$, the velocities of 2 and 3 are antiparallel to the velocity of 1 and half its size. See Figure 1. This follows from the action minimization of the eight. At the isosceles time $t = -\frac{1}{12}T$, the velocity of 2 is vertical, pointing down, and the velocities of 1 and 3 are such that their tangent lines pass through 2. This follows from the three-tangents theorem and the angular momentum property, both of which are described below.

Angular momentum and star-shapedness. Write

$$\ell_j = q_j \wedge \dot{q}_j$$

for the angular momentum of the j-th particle. Action minimization of the eight implies that its total angular momentum is zero:

$$\ell_1 + \ell_2 + \ell_3 = 0$$

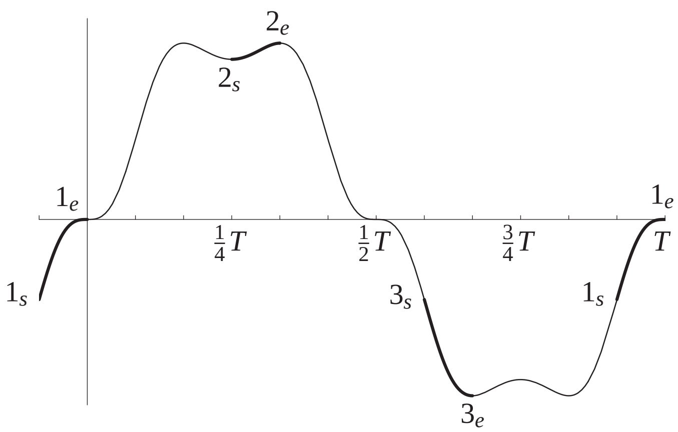

Figure 2. $\ell(t)$ versus t.

of the eight. Newton's equations imply (see [Chenciner and Montgomery 2000, p. 896])

$$\dot{\ell}_3 = \left(\frac{1}{r_{13}^3} - \frac{1}{r_{23}^3}\right)(q_1 \wedge q_2)$$

for all time. Upon taking account the distance inequality (2) and (3) we find that $\dot{\ell}_3 < 0$ on the arc 3. Similarly,

$$\dot{\ell}_1 > 0, \qquad \dot{\ell}_2 > 0, \qquad \dot{\ell}_3 < 0.$$

We use the notation 1_s to indicate body 1 at the starting time $t = -\frac{1}{12}T$, etc. By the symmetry, $\ell_{1_s} = \ell_{3_s} = -2\ell_{2_s} < 0$. (The inequalities $\ell_{1_s} < 0$ and $\ell_{1_e} = 0$ are consistent with $\dot{\ell}_1 > 0$.) Also $\ell_{2_s} > 0$ and $\dot{\ell}_2 > 0$ imply $\ell_{2_e} = -\ell_{3_e} > 0$. (See Figure 2.) Therefore over the interior $(-\frac{1}{12}T, 0)$ of our fundamental domain we have

$$\ell_1 < 0, \qquad \ell_2 > 0, \qquad \ell_3 < 0.$$

More generally, set

$$\ell = q \wedge \dot{q}$$

as q varies over the eight. It follows that on the right lobe $(x > 0)$ we have

$$\ell < 0 \quad \text{for } x > 0.$$

(See Figure 2.)

A curve in the plane is called *star-shaped* with respect to the origin if every ray from the origin intersects the curve at most once. For a smooth curve, this is equivalent to the assertion that, when written in polar coordinates as $(r(t), \theta(t))$, the function $\theta(t)$ is strictly monotone and does not vary by more than 2π. Since

$\ell = r^2\dot\theta$ the star-shapedness of a curve (such as one lobe of the eight) which lies in the half-plane $x > 0$ is thus equivalent to $\ell \neq 0$.

The three-tangents theorem. The following theorem can be found in [Fujiwara et al. 2003], where it was used to establish the existence of a choreographic three-body lemniscate for a non-Newtonian potential.

Theorem 2 (Three tangents). *Let $(q_1(t), q_2(t), q_3(t))$ be three planar curves whose total linear and total angular momentum are zero. Then the three instantaneous tangent lines to these three curves are coincident — they all three intersect in the same* (*time-dependent*) *point or are parallel.*

Proof. Fix the time t. Because $\dot q_1 + \dot q_2 + \dot q_3 = 0$, translating all the q_i in the same fixed direction does not change the condition of having zero angular momentum. So, without loss of generality, we can choose the origin to be the point of intersection of the tangent lines to q_1 and q_2 at time t. Because the point $q_1(t)$ lies along the line through the origin in the direction $\dot q_1$ we have $q_1(t) \wedge \dot q_1(t) = 0$. Similarly $q_2(t) \wedge \dot q_2(t) = 0$. But the total angular momentum is zero so we must have $q_3(t) \wedge \dot q_3(t) = 0$ which asserts that the line tangent to the curve of q_3 at t also passes through the origin. □

The proof also works for unequal masses m_1, m_2, m_3. Simply use the correct mass-weighted formulae for linear and angular momentum.

The splitting lemma. We will use the following splitting lemma in several places in the proof. A line in the plane divides the plane into three pieces: two open half-planes and the line itself. We say that a point lies *strictly on one side* of the line if it lies in one of the open half-planes. We say that this line *splits* the points A and B of the plane if the two points lie in opposite open half-planes.

Lemma 1. *Let $(q_1(t), q_2(t), q_3(t))$ be a planar solution to Newton's three-body equation with attractive $1/r$ potential. Suppose that at time t_* the arc $q_i(t)$ of mass i has an inflection point and nonzero speed. Then the tangent line ℓ to this arc at time t_* must either* (A) *split the other two masses $q_j(t_*)$ and $q_k(t_*)$ or* (B) *all three masses must lie on this tangent line.*

Proof. Suppose, to the contrary, that either both $q_j(t_*)$ and $q_k(t_*)$ lie strictly on one side of ℓ, or that one lies on ℓ while the other lies strictly on one side. According to Newton's equations the acceleration $\ddot q_i(t_*)$ is a linear combination of $q_j(t_*) - q_i(t_*)$ and $q_k(t_*) - q_i(t_*)$ and the coefficients of this linear combination are positive. Thus, translating ℓ and the configuration of masses back to the origin by subtracting $q_i(t_*)$, we see that this acceleration lies strictly on one side of the line through 0 spanned by the velocity $\dot q_i(t_*)$. Consequently, the acceleration and velocity of $q_i(t)$ are *linearly independent* at t_*. But the condition of being an inflection point is precisely that the acceleration and velocity be linearly dependent. □

The same proof works if the Newtonian potential $-\sum_{i<j} m_i m_j / r_{ij}$ is replaced by any potential $V = \sum_{i<j} f(r_{ij})$, where $df/dr > 0$.

A Convexity Proposition. A parametrization t of a curve C is *nondengenerate* if the derivative $dC(t)/dt$ is never zero. A smooth, possibly self-intersecting curve is called *locally convex* if its curvature never vanishes.

Proposition. *Let C be a smooth locally convex planar curve parametrized by a nondegenerate parameter t. Let $\ell(t)$ be the tangent to C at $C(t)$. Let m be a line not intersecting C. Let $P(t)$ be the point of intersection of $\ell(t)$ and m. Then $P(t)$ moves on the line m always in the same direction, for all t such that $P(t)$ is finite.*

Proof. We can take m to be the y-axis. If C is parametrized by $(x(t), y(t))$, the line $\ell(t)$ is given by $\{(x(t), y(t)) + \lambda(\dot{x}(t), \dot{y}(t)) : \lambda \in \mathbb{R}\}$, and it intersects m at $P(t) = (0, p(t))$, where

$$p = -\frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{\dot{x}(t)}.$$

Differentiation and the definition of the curvature κ yield

$$\frac{dp}{dt} = -\frac{v^3 x}{\dot{x}^2}\kappa,$$

where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ is the curve's speed. The factors v, x, κ are never zero by assumption (in the case of x because C avoids m); therefore they have constant sign. Thus dp/dt has constant sign wherever defined. □

3. To each mass its own quadrant

A crucial ingredient in the proof of Theorem 1 is that each mass "stays in its own quadrant" during the time interval $(-\frac{1}{12}T, 0)$. Initially 3 is in the first quadrant, 1 is in the fourth, and 2 is on the x-axis between the second and third quadrants, moving into the third. Hence, for a short time interval $(-\frac{1}{12}T, -\frac{1}{12}T + \epsilon)$, mass 3 lies in the first quadrant, 1 in the fourth, and 2 in the third.

Lemma 2. *Over the time interval $(-\frac{1}{12}T, 0)$ body 1 lies in the fourth quadrant, body 2 lies in the third, and body 3 lies in the first.*

Proof. Suppose one of the masses leaves its initial quadrant before time 0. It must exit along the boundary of this quadrant. It cannot exit through the origin, as this would imply an Euler configuration and the only Euler configuration occurs at the endpoint of the interval.

We argue individually that each mass cannot be the first to exit. Suppose that 2 exits first (perhaps simultaneously with another). It cannot leave crossing the x-axis, as this would contradict star-shapedness of the lobe it lies on. Neither can it exit through the y-axis, for then its x-coordinate would be zero, and, because collinearity of the three masses is excluded, at least one of 1 and 3 would not be exiting at the same time and so would have a positive x-coordinate. Thus the sum of the x-coordinates of the masses would be positive, contradicting that the center of mass is at the origin.

Mass 1 cannot leave first. For it cannot leave through the x-axis, as this would again contradict star-shapedness. It cannot leave through the y-axis as this would violate the distance ordering $r_{13} < r_{12} < r_{23}$ guaranteed by (2). To see this violation, write the exit point for mass 1 as $(0, y_1)$, with $y_1 < 0$. Then the other masses must be at $(-x, y_2)$ and (x, y_3) with $x > 0$ (since the configuration cannot be collinear) and $y_2 < 0$, $y_3 > 0$. We have $r_{13}^2 = x^2 + (y_3 - y_1)^2$ and $r_{12}^2 = x^2 + (y_2 - y_1)^2$. But $y_3 > 0,\ 0 > y_1, y_2$, and $y_1 + y_2 + y_3 = 0$, so

$$y_3 - y_1 = -2y_1 - y_2 = 2|y_1| + |y_2|,$$

while $|y_2 - y_1| < |y_2| + |y_1|$, so that $(y_3 - y_1)^2 > (y_2 - y_1)^2$ and $r_{13} > r_{12}$, contradicting the distance ordering.

Mass 3 cannot leave first. It cannot exit across the x-axis, for if it did the center of mass of the system would have a negative y-coordinate. It cannot leave across the y-axis, for this would contradict star-shapedness. □

4. Proof of Theorem 1

We refer to the arc swept out by mass j during the the time interval $\left[-\frac{1}{12}T, 0\right]$ as arc j, and write κ_j for its curvature. We must show that $\kappa_1 \leq 0$ with $\kappa_1 < 0$ for $t \neq 0$, that $\kappa_2 > 0$ and that $\kappa_3 < 0$.

Convexity of arc 1. We begin by showing that $\ddot{y}_1 > 0$ along arc 1. Since each mass stays in its own quadrant, we have $y_3 - y_1 > 0$; moreover $r_{13} < r_{12}$ by (2). Thus

$$\begin{aligned}\ddot{y}_1 &= (y_3 - y_1)/r_{13}^3 + (y_2 - y_1)/r_{12}^3 \\ &> (y_3 - y_1)/r_{12}^3 + (y_2 - y_1)/r_{12}^3 \\ &= -3y_1/r_{12}^3 > 0.\end{aligned}$$

Next we show that $\dot{y}_1 > 0$ along the arc. From the fact that $\ddot{y}_1 > 0$, it suffices to show that $\dot{y}_1 > 0$ at the initial point of arc 1, the isosceles point. By the three-tangents theorem and the fact that $\ell_1 < 0$ it follows that at the isosceles point $\dot{q}_1$ points from q_1 to the vertex q_2, so that $\dot{y}_1 > 0$.

We have seen that $\ell_1 < 0$ while $\dot{\ell}_1 > 0$ along the arc. Combining these inequalities, we see that $\dot{\ell}_1 \dot{y}_1 - \ell_1 \ddot{y}_1 > 0$ holds along the arc. On the other hand, expanding the angular momentum, we get $\dot{\ell}_1 \dot{y}_1 - \ell_1 \ddot{y}_1 = (x_1 \ddot{y}_1 - y_1 \ddot{x}_1)\dot{y}_1 - (x_1 \dot{y}_1 - y_1 \dot{x}_1)\ddot{y}_1 = y_1(\dot{x}_1 \ddot{y}_1 - \dot{y}_1 \ddot{x}_1) = y_1 v_1^3 \kappa$. Thus $y_1 v_1^3 \kappa_1 > 0$. Since $y_1 < 0$, $v_1 > 0$ we have $\kappa_1 < 0$.

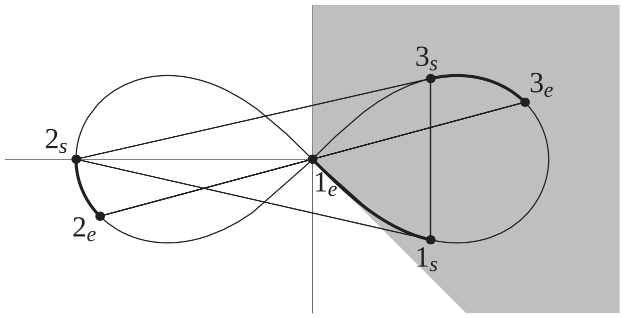

Figure 3. Region for bodies 1 and 3.

Convexity of arc 2. Assume, by way of contradiction, that there exists an inflection point $\kappa_2 = 0$ on arc 2. Let a be the last inflection point on arc 2 — the one whose time t is closest to 0. From the initial conditions at $t = -\frac{1}{12}T, 0$ described above we also know that $\kappa_2 > 0$ at the points 2_s and 2_e. By continuity, $\kappa_2 > 0$ near both of these points. Then $\kappa_2 > 0$ on the arc from a to 2_e.

We already know that arc 1 is convex ($\kappa_1 < 0$) and we also know that body 3 moves in the first quadrant. It follows that bodies 1 and 3 must lie within the shaded region in the Figure 3.

Consider the Gauss map (hodograph) of arc 2. This is the map that assigns to a point of arc 2 the unit tangent to arc 2, $\dot{q}_2/|\dot{q}_2|$, at that point.

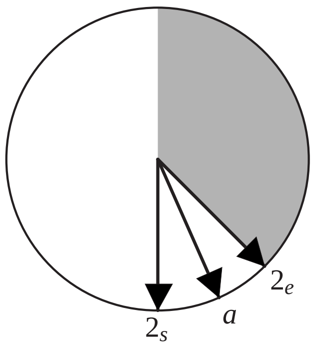

Figure 4. Gauss map of the unit tangent vector $\dot{q}_2/|\dot{q}_2|$.

By Newton's equation and the fact that $x_1 - x_2$ and $x_3 - x_2$ are positive we have $\ddot{x}_2 > 0$ on the entire arc 2. Since $\dot{x}_2 = 0$ at 2_s, this implies that $\dot{x}_2 > 0$ on the open arc of 2, from 2_s to 2_e, and so in particular $\dot{x}_2 > 0$ at a. Since $\kappa_2 > 0$ on the arc $a \to 2_e$, the vector $\dot{q}_2/|\dot{q}_2|$ must approach 2_e from the point a monotonically counterclockwise. Therefore the point a lies on the arc between the points 2_s and 2_e on the right half of the circle as shown in the Gauss map (Figure 4).

But then the tangent line to arc 2 at a cannot split the points 1 and 3, which, according to the splitting lemma (Lemma 1), contradicts the assumption that a is an inflection point.

Thus we have proved that arc 2 has no inflection points, that is, $\kappa_2 > 0$.

Convexity of arc 3. Assume, by way of contradiction, that there are inflection points on arc 3. Let b be the first such point, the one for which the time t is closest to $-\frac{1}{12}T$. Then, by the splitting lemma (Lemma 1), the tangent line to arc 3 at b must split bodies 1 and 2. In order to do that, the line must have passed earlier through either body 1 or body 2. We argue that both passings are impossible.

The tangent line to arc 3 cannot pass through body 1. For, by the three-tangent theorem, at the instant this happened, the tangent line from the body 2 would also pass through the body 1. We have already proved that $\kappa_2 > 0$ on the arc 2. Thus the tangent line from the body 2 never pass through the body 1 in this interval. (See Figures 3 and 4.) This is a contradiction.

The tangent line to arc 3 cannot pass through body 2. For if it did, by the three-tangents theorem, the tangent line to 1's curve would also pass through body 2 at the same instant. To see that this latter passing is impossible, join the endpoints 2_s and 2_e of arc 2 by a straight line m. Arc 2 lies completely on one side of this line, by convexity.

We now apply Proposition on page 192 to our situation. At the final points 1_e and 2_e, the tangents to 1 and 2 are parallel, so that the intersection of m with 1's tangent lies in the massless quadrant $x < 0$, $y > 0$. At the initial point s the intersection point of m and arc 1's tangent is 2_s. Consequently, in between s and e the intersection always lies in that part of m lying in the massless quadrant. But in order for 1's tangent to pass through 2, 1's tangent would have to cross line m between 2_s and 2_e, which is in the quadrant of arc 2, and hence it is impossible that this tangent passes through 2.

Therefore, we have proved that there is no inflection point on the arc 3. In other word, $\kappa_3 < 0$ on the arc 3.

Putting together the convexity of all three arcs we obtain Theorem 1.

5. Convexity for other potentials

Theorem 1 holds for the figure eight solution of other potentials. Indeed, our proof only depended on the properties of the eight listed in Section 2 and a monotonicity property of the Newtonian potential discussed below.

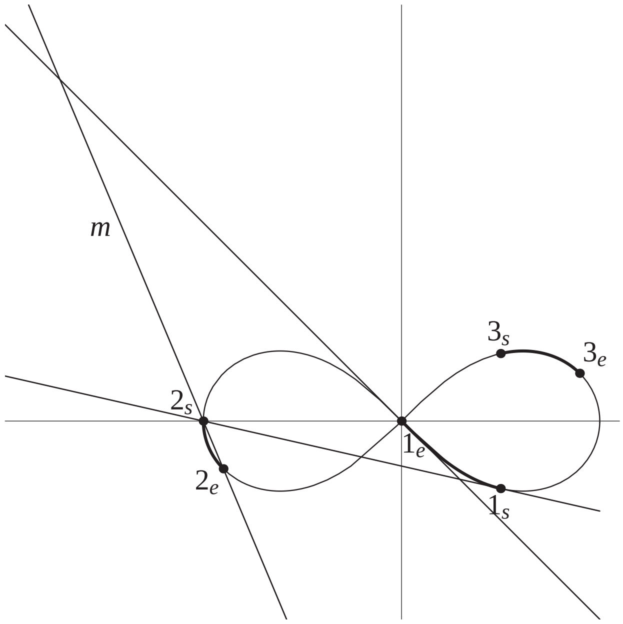

Figure 5. Line m and tangent lines to arc 1 at $t = -\frac{1}{12}T$ and $t = 0$.

To be precise, we need to define what we mean by an eight. Let

$$V = V(r_{12}, r_{23}, r_{31})$$

be a three-body potential depending only on the interparticle distances r_{ij} and invariant under interchange of the masses. Then the symmetry group D_6 of the eight acts on solutions to the corresponding Newton equation, taking solutions to solutions, and so we can speak of D_6-*invariant solutions*.

A planar solution to the Newton's equation for V is called an *eight solution* if

(i) it is invariant under the D_6 symmetries,

(ii) on the interior of each fundamental domain $\left(m\frac{1}{12}T, (m+1)\frac{1}{12}T\right)$, for $m = 0, \pm 1, \pm 2, \ldots$, the configuration is never collinear and never isosceles, and

(iii) the solution has no collisions.

Such a solution will necessarily be a planar choreography (see (1) on page 188), and so the three masses travel a single planar curve. Condition (i) implies that the center of mass is 0 and that the angular momentum is zero. If, in addition, our potential V has the form

$$V = \sum_{i<j} f(r_{ij}),$$

where

(iv) $df/dr > 0$ (attractive two-body potential) and

(v) $g(r) := r^{-1}df/dr$ is a strictly monotone decreasing function of r,

then all properties and inequalities used in this paper hold.

Indeed, return to the starting point, the distance ordering inequality (2). At $t = -\frac{1}{12}T$ we have $r_{23} = r_{12}$, and at $t = 0$ we have $r_{12} = r_{31} < r_{23} = 2r_{12}$. By property (ii), the possible distance orderings on the time interval $\left(-\frac{1}{12}T, 0\right)$ are $r_{31} < r_{12} < r_{23}$ or $r_{12} < r_{31} < r_{23}$. Consider the equation for $\dot{\ell}_1$,

$$\dot{\ell}_1 = \big(g(r_{21}) - g(r_{31})\big)(q_2 \wedge q_3),$$

for a monotone decreasing function $g(r)$. We have $\dot{\ell}_1 > 0$ for the first ordering and $\dot{\ell}_1 < 0$ for the second ordering. But, since $\ell_1 < 0$ at $t = -\frac{1}{12}T$ and $\ell_1 = 0$ at $t = 0$, the value of $\dot{\ell}_1$ must be positive. So we must have the first ordering, namely, equation (2). Then all equalities and inequalities in this paper hold. Thus:

Theorem 3. *Let V be a three-body potential of the form $V = \sum_{i<j} f(r_{ij})$ where f satisfies (iv) and (v) above, and admitting an eight solution as defined by (i)–(iii) above. Then each lobe of this eight for V is convex.*

The theorem begs the question, do eight solutions exist for any potentials besides Newton's? Recall from [Chenciner and Montgomery 2000, pp. 896–897] that if a solution that satisfies (i) and (ii) is known to minimize the action associated to V among all paths satisfying (i), and if that solution is not identically collinear, then automatically the solution satisfies (ii). The power law potentials

$$V_a = (a)^{-1}(r_{12}^a + r_{23}^a + r_{31}^a),$$

for $a \le -2$ admit such collision-free action minimizing solutions, and consequently they admit eight solutions. Moreover, the proof of [Chenciner and Montgomery 2000], specific to $a = -1$, is based on strict inequalities, and hence is valid for a range of exponents $-1 - \epsilon_1 < a < -1 + \epsilon_2$ for ϵ_1, ϵ_2 positive numbers. Numerical evidence presented in [Chenciner et al. 2002] suggests that eights exist for all power laws V_a, where $a < 0$. (These eights are dynamically stable only in a neighborhood of the Newtonian potential $a = -1$.)

Corollary. *For the power law potentials V_a with $a \le -2$ or with a in some open interval about -1, there exist eight solutions and each lobe of these eight solutions is convex.*

6. Unicity

Showing the unicity of the Newtonian eight remains an open problem [Chenciner 2003]. Our work here drastically reduces the candidate eights, and hence the scope of nonunicity, to those eights with convex lobes. It might allow a handhold towards surmounting the unicity problem. If our reader will allow us to fantasize in this direction, imagine two distinct Newtonian eights, both enjoying (i) D_6 symmetry, (ii) the same period, and (iii) having the same minimum value for the action. Connect these two eights by a family of eights having (i) and (ii), and having convex lobes. Apply the min-max procedure to extract out of such a family a third eight that is variationally unstable, meaning that the Hessian of the action there has a negative direction. Now establish a contradiction between the existence of the negative mode and the convexity of the lobe of this third eight. Such a program, or a similar one, could conceivably lead to a proof of unicity of the eight.

Acknowledgments

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Note added in proof

For the power law potentials V_a, Barutello, Ferrario and Terracini [Barutello et al. 2004] have proved existence of eights for all $a < 0$; see the proof following Proposition (4.15) on p. 19. Montgomery [2004] has proved the uniqueness of the eight for $a = -2$.

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