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my copy Letters in Math. Phys. v. 8, 1984 p59-67.

# CANONICAL FORMULATIONS OF A CLASSICAL PARTICLE IN A YANG–MILLS FIELD AND WONG'S EQUATIONS

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ABSTRACT. Wong [14] introduced equations of motion for a spin 0 particle in a Yang–Mills field which were widely accepted among physicists. It is shown that these are equivalent to the various mathematical formulations for the motion of such particles as given by the Kaluza–Klein formulation of Kerner [4], and those of Sternberg [11], and Weinstein [12]. In doing this, we show that Sternberg's space is, in a natural way, a symplectic leaf of a reduced Poisson manifold and relations to a construction of Kummer's [5] for dynamics on the cotangent bundle of a principle bundle are clarified.

## 1. INTRODUCTION

Wong's equations are

$$\dot{p}_\mu = F^a_{\mu\nu}\xi_a p^\nu - \frac{1}{2}\frac{\partial g^{\alpha\nu}}{\partial x^\mu}p_\alpha p_\nu, \tag{1a}$$

$$\dot{\xi}_a = -c^d_{ab}A^b_\mu p^\mu \xi_d, \tag{1b}$$

where $g$ is a Lorentz-metric on a space-time $X$. $A$ is a connection on a principal bundle $P$ over $X$. $F$ is its curvature, $\xi_a$ are a basis for the Lie algebra $\mathfrak{G}$ of the structure group $G$ of $P$, the $c$'s are the structure constants of $\mathfrak{G}$, $x^\mu$ are space-time coordinates, and $p_\mu$ are the resulting momentum (cotangent) coordinates. Space-time indices are raised and lowered by $g$. We will assume there is a bi-invariant metric $\gamma$ on $\mathfrak{G}$ with which Lie algebra indices can be raised and lowered. These equations are amended by

$$\dot{x}^\mu = p^\mu \tag{1c}$$

and the interpretation $\dot{} = m(\mathrm{d}/\mathrm{d}\tau)$, where $m$ is the particle's rest mass, and $\tau$ its proper time. For convenience the coupling constant and Planck's constant have been set equal to one.

The geometric interpretation of these equations is as follows. Equation (1a) is that for the worldline of a particle under a generalized Lorentz force. Equation (1b) says that the isotropic spin momentum, which is a section of the associated co-adjoint bundle

*Letters in Mathematical Physics* 8 (1984) 59–67. 0377–9017/84/0081–0059 $01.35.

$$E = P \times_G \mathfrak{G}^* \text{ (with fibre coordinates } \xi_a)$$

over the wordline, is parallel translated by the connection induced on $E$ by $A$.

There are three other formulations of the dynamics of a particle in a Yang–Mills field. The Kaluza–Klein picture of electromagnetism was given a straightforward generalization to the Yang–Mills case by Kerner [4]. Sternberg [11] gave the first formulation in the spirit of the modern school of symplectic geometers. Lastly, soon after Sternberg, Weinstein [12] gave a natural formulation using reduction and showed it was equivalent to Sternberg's. Sniatycki [10] showed that Kerner's and Weinstein's, hence Sternberg's, formulations predict the same world-lines on space-time. A straightforward calculation done in Section 2 shows that this worldline is also given by Wong's equation (1a).

It has been unclear, however, how the various formulations are related in the fibre direction, that is, to Wong's equation (1b). In this paper we show, that all formulations are equivalent in a natural way (Theorems 1 and 2).

A guiding principle for this work has been that Wong's equations naturally live on the vector bundle $E^{\#} = P^{\#} \times_G \mathfrak{G}^*$ and not on the submanifold of $E^{\#}$ given by $\mathcal{O}^{\#} = P^{\#} \times_G \mathcal{O}$, which is Sternberg's phase space. Here $P^{\#}$ denotes the pullback of $P$ obtained by completing the following diagram with dotted arrows

$$\begin{array}{ccc} P^{\#} & \dashrightarrow & P \\ \downarrow & & \downarrow \\ T^*X & \longrightarrow & X \end{array}$$

and $\mathcal{O}$ is a co-adjoint orbit in $\mathfrak{G}^*$. The key to the equivalence of the various formulations was the realization that $A$ induces a Poisson structure on $E^{\#}$ whose symplectic leaves are the $\mathcal{O}^{\#}$'s (see Theorem 2).

## 2. KALUZA–KLEIN AND KERNER

In the Kaluzu–Klein picture of the motion of a colored particle in a Yang–Mills field, as generalized by Kerner, the motion of the particle is a geodesic on $P$ where the metric $K$ on $P$ is induced by the connection $A$. That is,

$$T_pP = V_p \oplus H_p, \quad p \in P \tag{2}$$

is an orthogonal decomposition, where $V_p$ is the vertical space and $H_p$ is the connection's horizontal distribution. On $V_p$, $K$ is induced by the fixed bi-invariant metric $\gamma$ on $\mathfrak{G}$ under the infinitesimal generator isomorphism

$$\sigma_p\colon \mathfrak{G} \to V_p \subseteq T_pP \tag{3}$$

and on $H_p$ it is the horizontal lift of the metric $g$. on space-time, via the connection.

The metric $K$ is $G$-invariant. Its geodesics are determined by the $G$-invariant kinetic Hamiltonian on $T^*P$ where the symplectic structure is the standard one.

## 3. THE EQUIVALENCES

*Notation*: There will be many bundles in the discussions below. To ease the notational burden we will use the symbol $\pi_{YZ}$ for a bundle projection $Z \to Y$. When no problems arise from it, the subscript $YZ$ will be dropped. If $Z$ is a product we may use $\pi_1$ and $\pi_2$ instead for projection onto the first and second factors. Also $\tau_Y$ will always be used for the cotangent projection $T^*Y \to Y$.

We begin by decomposing $T^*P$ into a horizontal part which turns out to be $P^\#$, and a vertical part $\mathfrak{G}^*$, by using the connection $A$. To do this, we use an observation of Guillemin and Sternberg's [3] that $P^\#$ has a natural realization as $V^0 \subseteq T^*P$, the annihilator of the vertical bundle $V \subseteq TP$. Then the projection $P^\# \to P$ is just the restriction of the cotangent projection $\tau_P$, and the other projection $P^\# \to T^*X$ is defined by

$$\pi_{T^*X,\, P^\#}(\alpha_p) \cdot v = \alpha_p \cdot V$$

where $V$ is any vector in $T_pP$ which projects to $v$, that is such that

$$T\pi_{XP} \cdot V = v \quad \text{and} \quad \alpha_p \in P^\#_p \subseteq T^*_pP.$$

The connection $A$ induces a $G$-equivariant isomorphism of vector bundles over $P$:

$$\widetilde{A}: p^\# \times \mathfrak{G}^* \xrightarrow{\simeq} T^*P$$

given by

$$\widetilde{A}(\alpha_p, \mu) = \alpha_p + A^*_p\mu$$

where $A^*_p$ is the dual of the connection form $A_p: T_pP \to \mathfrak{G}$. This is just the splitting of $T^*P$ dual to the splitting (2) of $TP$.

Using $\widetilde{A}$, we pull back the canonical two-form and the Kaluza–Klein Hamiltonian to get a $G$-invariant Hamiltonian system on $P^\# \times \mathfrak{G}^*$. If we quotient this by $G$ we have a Poisson structure (see Marsden and Weinstein [7, 8] or Weinstein, [13] and Hamiltonian $H^\#$ on the associated vector bundle to $P^\#$: $E^\# = P^\# \times_G \mathfrak{G}^*$. This is the phase space for Wong's equations.

THEOREM 1. *Hamilton's equations on $E^\#$ are Wong's equations.*

THEOREM 2. *The Sternberg phase spaces $\mathcal{O}^\# = P^\# \times_G \mathcal{O} \subseteq E^\#$, where $\mathcal{O} \subseteq \mathfrak{G}^*$ is a coadjoint orbit, are the symplectic leaves of $E^\#$. Sternberg's Hamiltonian, considered as a function on $E^\#$, differs from ours by a Casimir function (one constant on symplectic leaves) and hence, generates the same equations of motion.*

REMARK. Weinstein [12] has constructed an equivalence between his set-up and Sternberg's. Hence, Theorems 1 and 2 together show all four formalisms are equivalent.

## 4. PROOF OF THEOREM 1

We have the commutative diagram

$$\begin{array}{ccc} P^{\#} \times \mathfrak{G}^* & \xrightarrow{\tilde{A}} & T^*P \\ \downarrow & \swarrow & \downarrow \\ E^{\#} = P^{\#} \times_G \mathfrak{G}^* & \xrightarrow[\bar{A}]{} & T^*P/G \\ & \searrow \quad \swarrow & \\ & T^*X & \end{array} \qquad \text{(CD1)}$$

in which all maps except the bottom two are, by definition of the Poisson structures, Poisson maps. The idea is to put coordinates on $E^{\#}$ corresponding to those in Wong's equations, lift these up to $T^*P$ and compute brackets there.

Let $U$ be an open subset of $X$ with coordinates $x^{\mu}$. Let $s: U \to P$ be a local section and $\phi: U \times G \simeq P_U$ the corresponding local trivialization.

$$T^*\phi: T^*P_U \simeq T^*U \times T^*G \simeq T^*U \times G \times \mathfrak{G}^* \qquad \text{(3a)}$$

where the second isomorphism is *right* trivialization of $T^*G$. Under this isomorphism $P^{\#}$ consists of those elements whose $\mathfrak{G}^*$ factor is zero. Thus

$$P^{\#}_U \times \mathfrak{G}^* \simeq (T^*U \times G) \times \mathfrak{G}^* \qquad \text{(3b)}$$

$$E^{\#}_U \sim T^*U \times \mathfrak{G}^*. \qquad \text{(3c)}$$

The decompositions of $P^{\#}_U$ and $E^{\#}_U$ are local trivializations in the fibre bundle sense as fibre bundle over $T^*X$. This is not so for the decomposition of $T^*P_U$.

Composing the maps $T^*P \to T^*P/G \to T^*X$* in order to eliminate the $T^*P/G$ vertex of (CD1), the local version of this diagram becomes

$$\begin{array}{ccc} (x, a, g, \mu) & \xmapsto{\tilde{A}} & (x, a + \mathrm{A}(x)^* \mathrm{Ad}^*_{g^{-1}}\mu, g, \mathrm{Ad}^*_{g^{-1}}\mu) \\ \downarrow & & \downarrow \\ (x, a, \mathrm{Ad}^*_{g^{-1}}\mu) & \longmapsto & (x, a) \end{array} \qquad \text{(CD2)}$$

where $a \in T^*_xX$, and $\mathrm{A}(x) = s^*A(x)$. This is seen by noting that in the local trivialization $\phi$ that $A$

*This composition is the map Weinstein uses in [12], namely, the dual of horizontal lift $T_xX \to T_pP$.

is given by

$$A(x, g) \cdot (v, \xi) = \mathrm{Ad}_{g^{-1}} \mathrm{A}(x) \cdot v + \mathrm{Ad}_{g^{-1}} \xi$$

where $(v, \xi) \in T_x U \times \mathfrak{G}$. Here we have used the decomposition

$$T\phi \colon TP_U \simeq TU \times TG \simeq TU \times G \times \mathfrak{G}$$

with the second isomorphism right trivialization of $TG$. This is the decomposition dual to that of $T^*P_U$. Then

$$A(x, g)^* = \mathrm{A}(x)^* \mathrm{Ad}_{g^{-1}} \oplus \mathrm{Ad}^*_{g^{-1}} : \mathfrak{G}^* \to T^*_x U \times \mathfrak{G}^*.$$

Now let $x^\mu$, $p_\nu$ be standard cotangent coordinates on $T^*U$ induced by the coordinates $x^\mu$ on $U$, and let $\xi_a$ be a basis for $\mathfrak{G}$, hence linear coordinates on $\mathfrak{G}^*$. Abusing notation, we consider $x^\mu, p_\nu, \xi_a$ as local functions on $T^*P$ on $E^\#$ by composing them with the projections onto $T^*U$ or $\mathfrak{G}^*$ defined by the local product structures (4a) and (4c). Note that $x^\mu$, $p_\nu$, $\xi_a$ are coordinates for $E^\#$ with $\xi_a$ the fibre coordinates and the projection $E^\# \to T^*X$ given by $(x^\mu, p_\nu, \xi_a) \mapsto (x^\mu, p_\nu)$.

If $f$ is a function on $E^\#$ we will let $\bar{f}$ denote the pullback of $f$ to $T^*P$ via the dotted diagonal in (CD1). One then reads off of (CD2)

$$\overline{x^\mu} = x^\mu, \qquad \overline{p_\nu} = p_\nu - A^a_\nu \xi_a, \qquad \overline{\xi_a} = \xi_a.$$

Here $\mathrm{A}(x) = A^a_\mu \, \mathrm{d}x^\mu \otimes \xi_a$ so $\mathrm{A}(x)^* = A^a_\mu \xi_a \otimes \mathrm{d}x^\mu$. One finds

$$H = \tfrac{1}{2}[g^{\mu\mu'} \bar{p}_\mu \bar{p}_\nu + \gamma^{ab} \xi_a \xi_b] \tag{4a}$$

is the Kaluza–Klein Hamiltonian on $T^*P$. It induces

$$H^\# = \tfrac{1}{2}[g^{\mu\nu} p_\mu p_\nu + \gamma^{ab} \xi_a \xi_b] \tag{4b}$$

on $E^\#$. Hamilton's equations $\dot{f} = \{\bar{f}, H^\#\}_\#$ on $E^\#$ are, by definition of the bracket $\{\,,\}_\#$ on $E^\#$, $\dot{f} = \{\bar{f}, H\}^\#$, where $\{\,,\}$ is the bracket on $T^*P$ and the superscript # means push the resulting $G$-invariant function back down to $E^\#$.

To get Wong's equations of motion we substitute the coordinate functions $x^\mu$, $p_\nu$, $\xi_a$, for $f$ in (4). We first calculate their brackets with each other. Since $x^\mu$ and $p_\mu$ are canonically conjugate on $T^*P$ we have

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu = \{\bar{x}^\mu, \bar{p}_\nu\} \tag{5a}$$

where in the second equality we have used the fact that

$$\{x^\mu, \xi_a\} = 0 = \{\bar{x}^\mu, \bar{\xi}^a\}. \tag{5b}$$

Also,

$$\{p_\mu, \xi_a\} = 0 \tag{5c}$$

and

$$\{\xi_a, \xi_b\} = c^d_{ab}\xi_d \tag{5d}$$

as is seen by noting that the $\xi_a$ are the components of the momentum map for the *left* action of $G$ on the $T^*G$ factor of $T^*P_U$ as decomposed in (4a), which is projection onto the $\mathfrak{G}^*$ factor of (4a). This is a Poisson map into $\mathfrak{G}^*_+$, that is, $\mathfrak{G}^*$ with its *right* Lie–Poisson structure. (For a good discussion of the right and left Poisson structures on the dual of a Lie algebra, see Marsden *et al.* [9]. Thus

$$\begin{aligned}\{\bar{p}_\mu, \bar{p}_\nu\} &= -\{p_\mu, A^a_\nu\xi_a\} - \{A^b_\mu\xi_b, p_\nu\} + A^a_\mu A^b_\nu\{\xi_a, \xi_b\} \\ &= A^a_{\nu,\mu}\xi_a - A^b_{\mu,\nu}\xi_b + c^d_{ab}A^a A^b_\nu\xi_d \\ &= F^a_{\mu\nu}\xi_a,\end{aligned} \tag{5e}$$

where $F$ is the curvature of the connection $A$. Finally

$$\{\xi_\mu, \bar{p}_\mu\} = -\{\xi_a, A^b_\mu\xi_b\} = -c^d_{ab}A^b_\mu\xi_d. \tag{5f}$$

Thus the equations of motion on $E^\#$ are

$$\dot{x}^\mu = \{\bar{x}^\mu, H\}^\# = \tfrac{1}{2}\{x^\mu, g^{\alpha\beta}\bar{p}_\alpha\bar{p}_\beta\}^\# = \bar{p}^{\mu\#} = p^\mu,$$

$$\begin{aligned}\dot{p}^\mu = \{\bar{p}^\mu, H\}^\# &= \tfrac{1}{2}[g^{\alpha\nu}(\{\bar{p}_\nu, \bar{p}_\alpha\}\bar{p}_\nu + \{\bar{p}_\mu, \bar{p}_\nu\}\bar{p}_\alpha) + \{\bar{p}_\mu, g^{\alpha\nu}\}\bar{p}_\alpha\bar{p}_\nu]^\# \\ &= F^a_{\mu\nu}\xi_a p^\nu - \frac{1}{2}\frac{\partial g^{\alpha\nu}}{\partial x^\mu}p_\alpha p_\nu,\end{aligned}$$

$$\begin{aligned}\dot{\xi}_a &= \{\bar{\xi}^a, H\}^\# = \tfrac{1}{2}[g^{\mu\nu}(\{\xi^a, \bar{p}_\mu\}\bar{p}_\nu + \{\xi^a, \bar{p}_\nu\}\bar{p}_\mu) + \gamma^{cd}(\{\xi_a, \xi_c\}\xi_d + \{\xi_a, \xi_d\}\xi_c)]^\# \\ &= -c^d_{ab}A^b_\mu\xi_d p^\mu + c^d_{ac}\xi_d\xi^c \\ &= -c^d_{ab}A^b_\mu\xi_d p^\mu.\end{aligned}$$

(The term $c^d_{ad}\xi_d\xi^c = 0$ because it equals $\Sigma_d\gamma([\xi_a, \xi_c], \xi_c) = -\Sigma_c\gamma(\xi_a, [\xi_c, \xi_c]) = 0$.) These are Wong's equations.

PROOF OF THEOREM 2

: have the following momentum maps $\Phi$ and $\pi_2$ for the *right* $G$ actions:

$$\begin{array}{ccc} P^{=} \times \mathfrak{G}^* & \xrightarrow{\;\widetilde{A}\;} & T^*P \\ & {\scriptstyle \pi_2} \searrow \quad \swarrow {\scriptstyle \Phi} & \\ & \mathfrak{G}^* & \end{array}$$

e $\Phi = \sigma^*$ where $\sigma_p : \mathfrak{G} \to T_pP$ is the infinitesimal generator, since the action of $T^*P$ is lifted n $P$. Let $\mathcal{O}$ be a co-adjoint orbit in $\mathfrak{G}^*$. Then $\widetilde{A}^{-1}\Phi^{-1}(\mathcal{O}) = P^{\#} \times \mathcal{O}$. From general theory ', for example, Marsden [6]), we know the reduced spaces $\Phi^{-1}(\mathcal{O})/G$ are the symplectic leaves $^*P/G$, and since $\bar{A}$ of (CD1) is a Poisson isomorphism, the same is true of $P^{\#} \times_G \mathcal{O} = \mathcal{O}^{\#}$, the nberg spaces.

'o see that the symplectic structure on $\mathcal{O}^{\#}$ is the same as the one Sternberg gives it, recall that reduced symplectic structure, $\omega^{\#}$ on $\mathcal{O}^{\#}$, which is the one it inherits as a symplectic leaf of is given by (Marsden [6], p. 31):

$$j^*\omega = \pi^*\omega^{\#} + \pi_2^*\omega_{\mathcal{O}^+} \tag{6}$$

re $\omega = \widetilde{A}^*\omega_P$ is the symplectic structure on $P^{\#} \times \mathfrak{G}^*$, $j: P^{\#} \times \mathcal{O} \hookrightarrow P^{\#} \times \mathfrak{G}^*$ is the inclusion, $^{\#} \times \mathcal{O} \to \mathcal{O}^{\#}$ the projection and $\omega_{\mathcal{O}^+}$ is the right (+) symplectic structure on the orbit. Stern- 's structure $\widetilde{\omega}_s$ is defined by

$$\widetilde{\omega} = \pi^*\omega_s \tag{7}$$

e

$$\widetilde{\omega} = \widetilde{\omega}_X + d\langle J, \pi^*_{P, P^{\#}}A \rangle + \pi_2^*\omega_{\mathcal{O}^-}.$$

$\widetilde{\omega}_x$ is the pullback of the canonical two-form $\omega_X = -d\theta_X$ on $T^*X$ by $P^{\#} \times \mathcal{O} \to P^{\#} \to T^*X$. $\pi_2$ is the momentum map for the *left* action on $\mathcal{O}$, and $\omega_{\mathcal{O}^-} = -\omega_{\mathcal{O}^+}$ is the left symplectic :ure on $\mathcal{O}$. Comparing (6), (7) and (8), we see that it suffices to show that

$$j^*\widetilde{A}^*\theta_P = (\pi_{T^*X, P^{\#} \times \mathfrak{G}^*})^*\theta_X - \langle J, \pi^*_{P, P^{\#}}A \rangle. \tag{8}$$

d, this may be verified by a straightforward calculation.

prove the final remark, recall that Sternberg's Hamiltonian $H_s$ is the pullback of the kinetic tonian on $T^*X$. In coordinates: $H_s = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$. This differs from the Kaluza–Klein tonian $H^{\#}$ (see (4b)) by the Casimir function $\frac{1}{2}\gamma^{ab}\xi_a\xi_b$.

.RK. As a curiosity, we note that the momentum lemma of Abraham and Marsden [1], p. : closely connected to Sternberg's construction. Indeed, in the case that the bundles are trivial, omentum lemma follows from Weinstein's isomorphism between his and Sternberg's construction.

In the introduction we interpreted solutions to Wong's equations as paths in the coadjoint bundle $E = P \times_G \mathfrak{G}^*$. We have seen that they are in fact constrained to $P \times_G \mathcal{O} \subseteq E$. This leads one to believe that the spaces on which we have formulated the equations, $E^\#$ or $\mathcal{O}^\#$, can be naturally embedded in the 'standard' phase spaces, $T^*E$ or $T^*(P \times_G \mathcal{O})$. For $\mathcal{O}^\#$ such an embedding is given by Kummer's construction [5].

Let $\mu \in \mathcal{O}$. Kummer has shown that a connection $\hat{A}$ on $P \to P/G$ induces an embedding of the reduced space $(T^*P)_\mu$ (of Marsden and Weinstein) as a symplectic subbundle of $T^*(P/G_\mu)$. Here $G_\mu$ is the isotropy group of $\mu \in \mathfrak{G}^*$, and the symplectic structure on $T^*(P/G_\mu)$ is noncanonical: it is the standard one $\omega_{P/G_\mu}$, minus the magnetic term $\tau^*_{P/G_\mu}(\mu \circ \hat{F})$ where $\hat{F}$ is the curvature of $\hat{A}$. (Note that $\mu \circ \hat{F}$ is a standard two-form on $P/G_\mu$ precisely because $G_\mu$ is $\mu$'s isotropy group.) But from general theory $(T^*P)_\mu \simeq \Phi^{-1}(\mathcal{O})/G$, and we have shown that our connection $A$ makes the latter space look like $\mathcal{O}^\#$. We connect these ideas here.

PROPOSITION. *Sternberg's phase space $\mathcal{O}^\#$ is embedded via Kummer's construction as the 'horizontal' subbundle of $T^*(P \times_G \mathcal{O})$.*

By 'horizontal' we mean the annihilator of the vertical-with-respect-to-the-projection $P \times_G \mathcal{O} \to$ subbundle of $T(P \times_G \mathcal{O})$.

To begin with, $P/G_\mu \simeq P \times_G G/G_\mu \simeq P \times_G \mathcal{O}$. The first isomorphism is $[p]_{G_\mu} \mapsto [p, 0]_G$ wher 0 is the identity coset in $G/G_\mu$. Thus, Kummer's result along with the remarks preceding the proposition, gives us a symplectic embedding $\mathcal{O}^\# \hookrightarrow T^*(P \times_G \mathcal{O})$ (the latter with the nonstandard symplectic structure) where we choose $\hat{A} = \mathrm{pr} \circ A$ with pr = projection $\mathfrak{G} \to \mathfrak{G}_\mu$ relative to the me

To see how $\mathcal{O}^\#$ is embedded, we follow the various maps: Let $p^\#$ denote an element in $P^\# \subseteq T$ Then

$$[p^\#, \mu]_G \mapsto [p^\# + A_p^*\mu]_{G_\mu} \mapsto [p^\#]_{G_\mu} \mapsto [(p^\#, (\mu, 0))]_G$$

$$\mathcal{O}^\# = P^\# \underset{G}{\times} \mathcal{O} \to (T^*P)_\mu \to T^*(P/G_\mu) \to T^*(P \underset{G}{\times} \mathcal{O})$$

where we have written elements over the spaces to which they belong. The first map is the map $\bar{A}: \mathcal{O}^\# \xrightarrow{\sim} \Phi^{-1}(\mathcal{O})/G$ of diagram (CD1) composed with the natural identification $\Phi^{-1}(\mathcal{O})/G \xrightarrow{\sim} \Phi^{-1}$ $= (T^*P)_\mu$. The second map is Kummer's embedding. The third is the isomorphism induced by the isomorphism $P/G_\mu \xrightarrow{\sim} P \times_G \mathcal{O}$.

Elements in the two cotangent bundles can be represented in the given form, because wheneve $P$ is a principal $K$ bundle, $T^*(P/K) \simeq \mathrm{Ker}\, J_K/K$ where $J_K$ is the dual of the infinitesimal generato that is, the momentum map for the lifted $K$ action on $T^*P$. This isomorphism $f$ is given by $\langle f([\alpha]_K), v\rangle = \langle \alpha, V\rangle$ where $\alpha \in \mathrm{Ker}_p J_K$, $v \in T_{[p]}P/K$ and $V \in T_pP$ projects to $v$. This formula shows that the image of $\mathcal{O}^\#$ is the horizontal bundle. Indeed, a vertical vector $v$ in $T_p(P \times_G \mathcal{O})$ is the projection of a vector $V = 0 \oplus v \in T_pP \oplus T_\mu \mathcal{O}$ and the final element in the sequence of maps, $[(p^\#, (\mu, 0))]$, is the equivalence class containing the co-vector $p^\# \oplus 0 \in T_p^*P \oplus T_\mu^*$ . Hence

$$\langle f[p^{\#}, (\mu, 0)], v \rangle = \langle p^{\#} \oplus 0, 0 \oplus \nu \rangle = 0$$

as claimed.

ACKNOWLEDGEMENT

I thank Ted Courant, Jerry Marsden, Tudor Ratiu, Jedrzej Sniatycki and Alan Weinstein for valuable discussions. The research was partially supported by DOE Contract DE-AT03-82ER12097.

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*(Received September 1, 1983)*