Exact identity for nonlinear wave propagation

Duncan Ralph,[1] Onuttom Narayan,[1] and Richard Montgomery[2]
[1]*Department of Physics, University of California, Santa Cruz, California 95064, USA*
[2]*Department of Mathematics, University of California, Santa Cruz, California 95064, USA*
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An exact identity for nonlinear sound wave propagation in one-dimensional media is shown to apply under a wide variety of conditions. The evidence suggests that this is true whenever there is a single input and output channel on each side of the medium, which can generally be achieved using monochromatic filters. It is shown mathematically that the identity *cannot* be derived from this condition and the known symmetries of the system. The question of what hidden symmetries in the system lead to the identity, and whether it applies even more generally to light waves, remains unresolved.

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The propagation of waves in nonlinear media is of great importance in a variety of fields, from seismology and fluid mechanics to biophysics. Most notably, they have been studied extensively [1–5] in communications. Despite their importance, exact results for nonlinear wave propagation are rare. Although the existence of solitons [6] demonstrates the possibility of exact and intrinsically nonlinear phenomena in wave propagation, often one is restricted to approximate perturbative results.

With monochromatic longitudinal waves incident on a quasi-one-dimensional dissipationless system, earlier numerical simulations and perturbation theory calculations yielded [7] the exact identity

$$J_{LR}(I) - J_{RL}(I) = 4J_{Th}(I). \tag{1}$$

Here $J_{LR}(I)$ and $J_{RL}(I)$ are the power transmitted from left (right) to right (left) when waves of intensity I impinge on the medium from the left (right) and $J_{Th}(I)$ is the net power flowing from right to left when half the incident intensity I falls on each side (with a random phase). Monochromatic filters at the incident frequency were placed at both ends of the system. They confined higher harmonics to the nonlinear medium, thereby keeping different frequency channels separate outside the system, of relevance in a communications context.

The identity in Eq. (1) is fundamentally nonlinear: both sides of the equation are zero in the linear regime. For a linear time-reversal invariant medium, the reciprocity theorem [8,9] ensures that the transmission coefficient is unaltered if the positions of the source and receiver are exchanged. Although time-reversal invariance can be broken by a magnetic field or by using magnetic materials [10] for light waves, this is not possible for sound [11]. Moreover, for a linear medium, superposition yields J_{Th}, which is zero if the medium is time-reversal invariant or (from the second law of thermodynamics) nondissipative.

Although remarkable, the identity in Eq. (1) was only obtained numerically in Ref. [7], with a limited set of configurations. It is important to ascertain how general it really is, which we proceed to do in this paper.

In Ref. [7], the propagation of waves was studied in chains of particles connected by nonlinear springs. The spring potentials were

$$V(y) = \frac{1}{2}y^2 + \frac{\epsilon}{4}y^4. \tag{2}$$

Here if y_i is the displacement of the ith particle from its equilibrium lattice position, then $y=y_i-y_{i-1}$ for the spring connecting the $i-1$th particle to the ith one. The parameter ϵ is a nonlinearity parameter. The disorder in the system came from the different masses $\{m_i\}$, but the potential $V(y)$ was the same for all the springs and was an even function of y. This is a disordered version of the Fermi-Pasta-Ulam [15] model.

The first and last particles also had filters attached, and were coupled to the external environment by the incident and outgoing waves. For a chain of length N, the equations of motion for the interior and terminal particles were

$$\begin{aligned} m_i\ddot{y}_i &= -\partial_{y_i}[V(y_i-y_{i-1}) + V(y_{i+1}-y_i)],\\ m_1\ddot{y}_1 &= -m_1\omega_0^2 y_1 - V'(y_1-y_2) - \kappa\dot{y}_1 + 2F_{L;inc}(t),\\ m_N\ddot{y}_N &= -m_N\omega_0^2 y_N - V'(y_N-y_{N-1}) - \kappa\dot{y}_N + 2F_{R;inc}(t). \end{aligned} \tag{3}$$

Here $F_{L;inc}(t)$ and $F_{R;inc}(t)$ are the forces exerted by the incident waves on the left and right boundaries (i.e., the first and Nth particles), respectively. As shown in Ref. [7], $-\kappa\dot{y}_1 + 2F_{L;inc}(t)$ is equal to $F_{L;inc}(t)+F_{L;out}(t)$, where $F_{L;out}(t)$ is the force exerted on the left boundary by the outgoing waves. Thus $-\kappa\dot{y}_1+2F_{L;inc}(t)$ is the total force on the left boundary from the exterior. A similar result can be obtained for the right boundary. The parameter κ can be obtained in terms of the wave velocity and bulk modulus of the medium outside the system and the cross sectional area of the system.

The parameter ω_0 is the frequency of the filters at the two ends, as we now show. If the incident waves are at a frequency ω_0 and $m_{1,N}$ are large, in the equations for the terminal particles, the mismatch between the component at frequency $n\omega_0$ of the left-hand side and the first term on the right-hand side is zero for $n=\pm 1$ and large otherwise. Therefore $y_{1,N}(\omega)\approx 0$ for higher harmonics of ω_0, while the components of the interior and exterior forces on the terminal particles (the last three terms in the equations) at the frequency ω_0 add up to zero. Thus the terminal particles are transparent to forces at frequency ω_0 and (immobile) reflect-

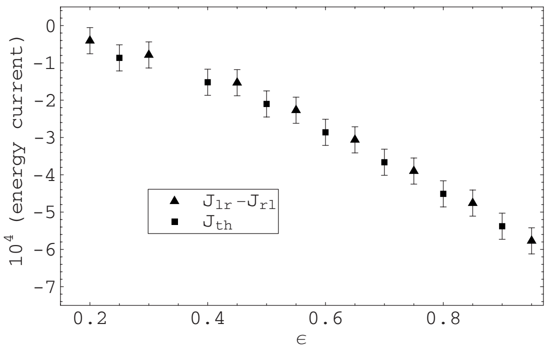

FIG. 1. Numerical results for $J_{LR}-J_{RL}$ and $4J_{Th}$ for a nearest neighbor interparticle potential Eq. (4). The spring constants are $\{c_i\}=\{1.0,0.9,0.95,1.15,1.1\}$. The masses of the particles are 50, 0.85, 0.7, 0.95, 0.65, and 50, and the damping parameter is $\kappa=0.5$ The incoming waves have a frequency $\omega_0=0.8$ and the intensity I is 0.25. Thus $2F_{L(R);inc}(t)=0.5\cos(0.8t)$ when the incident waves are from the left (right), and when they are from both sides, $2F_{L;inc}(t)=0.5/\sqrt{2}\cos(0.8t)$ and $2F_{R;inc}(t)=0.5/\sqrt{2}\cos(0.8t+\phi)$, with ϕ averaged over. The plot is as a function of ϵ, the strength of the nonlinearity in Eq. (4). As can be seen from the figure, Eq. (1) is valid even with this generalized nonlinear interaction.

ing boundaries at higher harmonics. The bandpass of the resultant filters $\delta\omega\sim\kappa/m_{1,N}$ can be made arbitrarily small but not zero.

Equation (1) was verified by simulating a chain with six particles. (Since the identity does not rely on the thermodynamic limit, short chains were adequate.) As the intensity of the incident waves was increased, a first order transition to a noisy state occurred, beyond which the Fourier transform of $y_i(t)$ exhibited broadband noise and Eq. (1) was violated. Equation (1) was also verified extremely accurately numerically using third order perturbation theory on a continuum model, and was seen to be invalid without the filters.

In the work reported here, the model used in Ref. [7] was extended in various ways to check the robustness of Eq. (1). The numerical molecular dynamics simulations used a fourth order Runge-Kutta algorithm, with a time step of $0.05\sqrt{2}$ to $0.15\sqrt{2}$. The initial positions and velocities of the particles were uniform random numbers with ranges comparable to their steady state values (from trial runs); it was verified that the results were insensitive to the initial conditions. The system was allowed to equilibrate for a time 10^4, after which measurements were taken over a similar time interval. The measurement interval was divided into segments, with the intersegment variation in measurements used to estimate the error.

In the first test, the interparticle potential of Eq. (2) was replaced with

$$V(y)=\frac{1}{2}(1-\epsilon c_i)y^2+\frac{\epsilon}{c_i}\exp(c_i y)-\epsilon y, \qquad (4)$$

where c_i was different for each spring and ϵ remains a nonlinearity parameter. A disorderless exponential potential [16–18] and a harmonic potential are integrable, but Eq. (7) is not. As shown in Fig. 1, Eq. (1) is valid independent of ϵ.

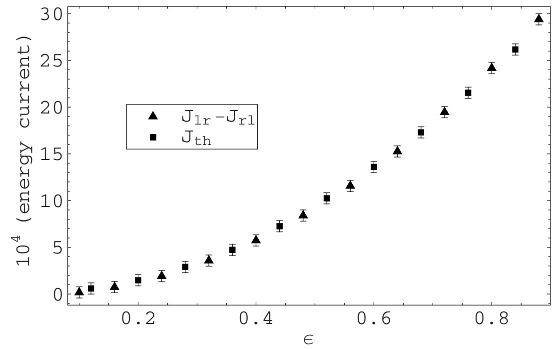

FIG. 2. Numerical results for $J_{LR}-J_{RL}$ and $4J_{Th}$ as a function of the nonlinearity ϵ. The filter frequencies were $\omega_{0,L}=0.8$ and $\omega_{0,R}=1.6$. All other system parameters and the interparticle potential are the same as in the previous figure, except that all the spring constants c_i's are unity. Despite the unequal filter frequencies, Eq. (1) is seen to be satisfied.

In the second test, the filters at the two ends of the system were adjusted so that their frequencies were ω_0 and $2\omega_0$, respectively, reminiscent of the configuration proposed in Ref. [19]. The interparticle potential was taken to be of the form given in Eq. (2). As seen in Fig. 2, Eq. (1) is satisfied even though the waves emerge from the system at different frequencies at the two ends.

Finally, the frequency of the incident waves was swept while adjusting the filter frequencies simultaneously and keeping all other system parameters constant. As shown in Fig. 3, Eq. (1) is satisfied over almost the entire range of frequency, except in a few narrow windows, discussed further later. Beyond $\omega_0\approx1.35$, the system chooses between multiple complicated steady states based on the initial conditions (Fig. 4). We have not analyzed this region exhaustively, but it seems consistent with the hypothesis that Eq. (1) is satisfied if and only if the outgoing waves are purely sinusoidal. For $\omega_0>1.6$, the transmission in both directions falls

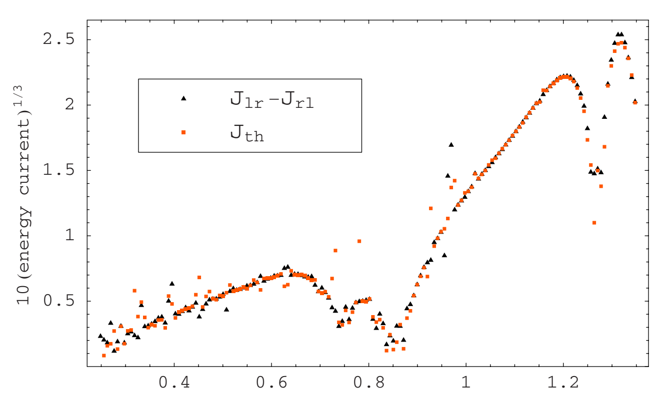

FIG. 3. (Color online) $J_{LR}-J_{RL}$ and $4J_{Th}$ as a function of the frequency of the incoming waves and the filters. (The magnitude of the cube root is plotted on the y axis to accommodate the large range of values.) The masses of the particles in the chain are (100,1.7,1.4,1.9,1.3,100). The damping parameter κ and the intensity I are 1. The interparticle potential is of the form in Eq. (2) with the nonlinearity parameter $\epsilon=0.08$. There are narrow windows at $\omega_0\approx0.73,0.96$, etc., where Eq. (1) is not satisfied.

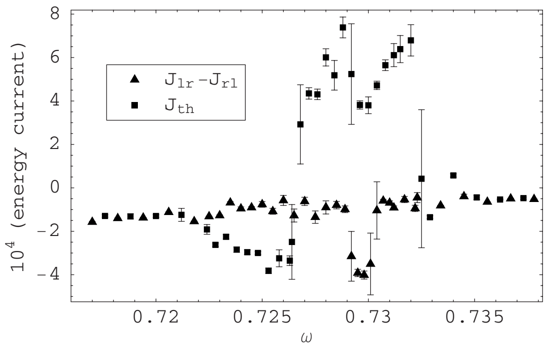

FIG. 4. An enlarged version of the window at $\omega_0 \approx 0.73$ from the previous figure.

off sharply; this is also beyond the frequencies of all the normal modes of the linearized system.

In view of the universality found for Eq. (1), its violation in narrow frequency windows is surprising and requires an explanation. As seen in Fig. 5, the primary oscillations at frequency ω_0 are amplitude modulated for $\omega_0 \approx 0.96$. The amplitude modulation is also seen in the displacement of the filter particles $y_{1,N}(t)$, and is therefore present in the outgoing waves emerging from the system.

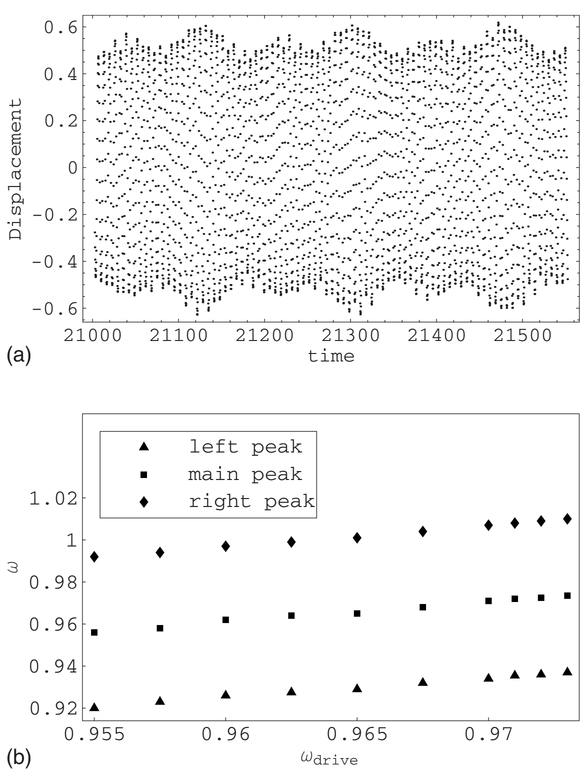

FIG. 5. Plot of $y_i(t)$ as a function of time for the particle at $i=3$ and at $\omega_0=0.96$. All the parameters of the system are the same as in the previous figure. Apart from the primary oscillations at the frequency ω_0, the plot shows amplitude modulation with a time period of $T \approx 180$, corresponding to a frequency of $\Omega \approx 0.0055$. In the lower panel, the frequency of the primary peak and the two adjoining side peaks in the Fourier transform of $y_i(t)$ are shown, demonstrating that Ω is reasonably constant across the window.

The amplitude modulated state can be understood as an example of parametric resonance. With $y_i(t)=\tilde{y}_i(t)+u_i(t)$, where $\tilde{y}_i(t)$ is the part of $y_i(t)$ that is periodic at the driving frequency ω_0, the linearized equation for $u_i(t)$ is

$$m_i\ddot{u}_i = (u_{i+1}+u_{i-1}-2u_i) + 3\epsilon[(\tilde{y}_{i+1}-\tilde{y}_i)^2(u_{i+1}-u_i) + (\tilde{y}_{i-1}-\tilde{y}_i)^2(u_{i-1}-u_i)]. \tag{5}$$

with appropriate modifications for the terminal particles. For small ϵ, we can calculate $\tilde{y}$ to zeroth order in ϵ, in which case $(\tilde{y}_{i+1}-\tilde{y}_i)^2=A_i+A_i\cos(2\omega_0 t+\phi_i)$. The second term on the right-hand side can trigger a parametric resonance at the frequency ω_0. We use the standard treatment for parametrically resonant systems [20]. If $u_i(t)$ is a periodic function at frequency ω_0 with a slowly varying amplitude, from Eqs. (5) the amplitude is found to grow as $\sim\exp[(\lambda+i\Omega)t]$, with $\lambda>0$ for approximately $0.955<\omega_0<0.98$. The growth of $u(t)$ is stabilized by terms nonlinear in u [neglected in Eq. (8)], resulting in an amplitude modulated steady state.

In view of the generality of Eq. (1), it is natural to attempt a mathematical proof that does not rely on the details of the system. The forces from the incident waves at the two ends of the system are the real parts of $z_{1,2}\exp[i\omega t]$, where $z_{1,2}$ are complex numbers with amplitude and phase information. (We assume that the filters at both ends are at equal frequencies; it is easy to extend this analysis.) Similarly, the outgoing waves at the edges of the system are $z'_{1,2}\exp[i\omega t]$. The system serves as a mapping $M:(z_1,z_2)\to(z'_1,z'_2)$. Without going into the details of the system [which we have seen do not affect the validity of Eq. (1)], what constraints are imposed on the mapping M by the symmetries and conservation laws of the system, and are these sufficient to yield Eq. (1)?

Time translation invariance requires that if $z_{1,2}\to z_{1,2}\exp[i\alpha]$, then $z'_{1,2}\to z'_{1,2}\exp[i\alpha]$. Therefore if $z_1=\rho_1\exp[i(\phi+\alpha)/2]$ and $z_2=\rho_2\exp[i(\alpha-\phi)/2]$, with similar expressions for the primed variables, then $\rho'_{1,2}$ and ϕ' do not depend on α. If we ignore the overall phases of the incoming waves and the outgoing waves, the system performs a mapping from (ρ_1,ρ_2,ϕ) to (ρ'_1,ρ'_2,ϕ'). Energy conservation [21] now requires that $\rho_1^2+\rho_2^2=\rho_1'^2+\rho_2'^2$. Therefore, if $\rho_1=r\cos(\theta/2)$ and $\rho_2=r\sin(\theta/2)$, with similar expressions for the primed variables, the mapping leaves r unchanged. For any r, one has a reduced mapping $M_r:(\theta,\phi)\to(\theta',\phi')$. Since all ϕ's are equivalent for $\theta=0,\pi$. the mapping M_r is from S^2 to S^2. The remaining condition on M_r comes from time-reversal invariance. Under time reversal, the incident and outgoing waves are reversed. The incident amplitude on the left is now the real part of $z'_1\exp[-i\omega t]$, which is equal to the real part of $z_1'^{*}\exp[i\omega t]$. Therefore time-reversal invariance implies that if $M(z_1,z_2)=(z'_1,z'_2)$, then $M(z_1'^{*},z_2'^{*})=(z_1^*,z_2^*)$. Defining $V=M_rC$, where $C(\theta,\phi)=(\theta,-\phi)$ is the complex conjugation operator, we have

$$V^2=I, \tag{6}$$

the identity operator.

Turning to Eq. (1), which we wish to prove, incident waves of intensity I from the left correspond to $\theta=0$ with $r=\sqrt{I}/2$ (see the caption to Fig. 1) and arbitrary ϕ, and $J_{LR}\propto I\sin^2\theta'/2$. Similarly, incident waves of the same intensity from the right correspond to $\theta=\pi$ with the same r, and $J_{RL} \propto I \cos^2 \theta'/2$. Therefore $J_{LR}-J_{RL} \propto -\frac{1}{2}I[\cos \theta_I'(0,\phi) +\cos \theta_I'(\pi,\phi)]$, where the subscript on θ' is a reminder that the mapping depends on I. Incident waves from both sides with intensity $I/2$ each and random relative phase correspond to $\theta=\pi/2$, with $r=\sqrt{I}/2$ and random ϕ. The net power flowing from right to left is proportional to $I \cos^2 \theta/2-I/2 =\frac{1}{2}I \cos \theta_I'(\pi/2,\phi)$. Equation (1) is then equivalent to

$$\cos \theta'(0,\phi) + \cos \theta'(\pi,\phi) = -4 \langle \cos \theta'(\pi/2,\phi)\rangle_\phi, \quad (7)$$

where the average on the right-hand side is over ϕ. This relates the image of the north and south poles under M_r to the image of the equator.

To show that Eq. (7) cannot be derived from Eq. (6), let V be a mapping that satisfies Eq. (6), and also satisfies a "separation property," that the images of the poles lie in "polar caps" $0<\theta'<\epsilon$ and $-\pi+\tilde{\epsilon}<\theta'<\pi$, in which the image of the equator never enters. The existence of such a mapping can be proved by an example: if V is a reflection about any great circle that is sufficiently close to the $\phi=0$ great circle but does not pass through the poles, then it satisfies Eq. (6) and the separation property. Let A be a mapping obtained by integrating the flow of a vector field that points southwards everywhere, but is zero inside the polar caps and on the equator. Since V satisfies Eq. (6), so does $U=AVA^{-1}$. If Eq. (7) is provable from Eq. (6), then V and U must satisfy Eq. (7). We now prove that if V satisfies Eq. (7), U does not.

A^{-1} acting on the north and south poles leaves them unchanged, V moves them to points within the polar caps, and A leaves them unchanged again. As a consequence, the mappings U and V have the same effects on the two poles, and the left-hand side of Eq. (7) is the same for V and U. On the right-hand side of this equation, A^{-1} leaves the equator unchanged, V maps it to a curve that lies outside the polar caps, and then A shifts this curve southwards. Therefore on the right-hand side of Eq. (7),

$$\cos \theta_U'(\pi/2,\phi) \leq \cos \theta_V'(\pi/2,\phi) \quad (8)$$

for any ϕ, where the subscripts U,V denote the mapping for which θ' is calculated. Therefore

$$\langle \cos \theta_U'(\pi/2,\phi)\rangle_\phi \leq \langle \cos \theta_V'(\pi/2,\phi)\rangle_\phi, \quad (9)$$

with the equality only valid if V maps the equator to itself (which is the trivial case of zero nonreciprocity).

Thus if V satisfies Eqs. (6) and (7), then U satisfies Eq. (6) but not Eq. (7). Moreover, for the specific example given above in which V is a reflection, $M_r(V)=VC$ can be isotoped to the identity since it is a rotation, as can A since it is the integral of a flow field. And if A is azimuthally symmetric, $M_r(U)=AVA^{-1}C=AM_r(V)CA^{-1}C=AM_r(V)A^{-1}$ can also be isotoped to the identity. Therefore Eq. (7) cannot be derived from Eq. (6) even with the additional condition that M_r should be smoothly connected to the identity operator. Therefore Eq. (1) cannot be derived from the known symmetries and conservation laws that apply to the system we have considered. In view of the robustness of Eq. (1) that we have found numerically, we conclude that there must be other hidden symmetries in the system that further constrain V and yield Eq. (1).

In this paper, we have shown that Eq. (1) is valid for longitudinal waves in a one-dimensional chain with monochromatic filters at the ends under very general conditions. The only restriction that we have found is that whenever, exploiting the finite bandpass of the filters, the outgoing waves are not monochromatic, the identity breaks down. Such a general result for nonlinear wave propagation, where exact results are difficult to obtain, is extremely important. We have also shown that the constraints and evident symmetries of the chain are not sufficient to yield Eq. (1). The question of what the hidden symmetries are, which when included, result in Eq. (1), and whether these symmetries are valid for light waves, is an open one.

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