Optimal Control of Deformable Bodies and Its Relation to Gauge Theory

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Abstract. I investigate the question "What is the most efficient way for a deformable body to deform itself so as to achieve a desired reorientation?" I call this the Cat's Problem, since itis the problem faced by the upside-down zero-angular momentum cat in freefall. In order of increasing generality, I show that the Cat's Problem is a special case of problems which occur (1) in the geometry of principal bundles, (2)in subRiemannian geometry, and (3) in optimal control. Some model cases are explicitly solved in which the deformable body consists of a collection of point masses. In one of these models the principal bundle breaks down due to isotropy for the action of the rotation group. Nevertheless we are still able to obtain the general solution.

§1. Introduction.

Consider the

Cat's Problem. What is the most efficient way for a deformable body to deform itself so as to achieve a desired rigid re-orientation?

A cat, dropped from upside-down with no angular momentum, changes her shape in such a way as to land on her feet. In doing so, her initial and final shape are essentially the same, but she has re-oriented herself by a rigid rotation of 180 degrees. In addition, by conservation of angular momentum, her total angular momentum is zero throughout the motion. For a nice analysis of this phenomenon, see Kane and Scher [**1969**]. The cat thus describes a loop in her shape space, with the consequence that in an inertial frame the beginning and final shapes are related by a rigid motion $g \in E(3)$.

Shapere and Wilczek addressed the cat's problem in [**1987**], [**1988**], where they translated it into a problem in gauge theory. Their original paper [**1987**] concerned the motion of microorganisms in water, so perhaps it is not fair to call it the cat's problem. In any case, Shapere and Wilczek observed the following key features of this problem.

(i) The space of inertial configurations, that is, the space of allowable deformations of the body, forms a principal bundle over the shape space of the body. The fiber of this bundle is the group G of rigid motions, an element of which is the desired re-orientation.

(ii) Dynamical constraints define a connection A on this bundle. For the cat in free-fall with no initial angular momentum this constraint is that the angular momentum remains zero. For the micro-organism the constraint is that the viscosity is infinite.

(iii) With the constraint in force, the holonomy (parallel translation operator with respect to the connection A) of a given sequence of shape changes <u>is</u> the net re-orientation.

(iv) Efficiency can be measured by a metric on shape space.

Consequently, the zero angular momentum cat problem becomes the **Isoholonomic problem.** Find the shortest loop in shape space (based at a given shape) with a given holonomy.

The cat's problem is more general because the cat may be spinning. That is, its angular momentum may be constrained to a non-zero constant μ. The main result of our paper, Theorem 1 in §3, is a characterization of the solutions to the cat's problem and hence to the isoholonomic problem. The characterization says that all solutions are obtained by solving Hamilton's differential equations for a certain Hamiltonian H_μ on the cotangent bundle of shape space. H_0 is the horizontal kinetic energy.

Our paper is based on the following two observations.

(I) The kinetic energy k, which is a Riemannian metric on the inertial configuration space Q, together with the action of the group G on Q, contain all the information in observations (i), (ii), and (iii).

(II) The isoholonomic problem is a special case of the problem of finding sub-Riemannian geodesics.

We elaborate on (I). (i) Shape space S is the quotient space Q/G. (ii) The connection A is determined by declaring the A-horizontal directions HOR to be those directions perpendicular to the group orbits. (iii) The metric on shape space is determined by declaring that the projection of the tangent space to Q onto the tangent space to shape space is an isometry when restricted to the horizontal space HOR. Concerning (II): a ***sub-Riemannian metric*** is the restriction of a Riemannian metric k to a distribution $HOR \subset TQ$. Sub-Riemannian metrics are also known as *Carnot-Caratheodory metrics, non-holonomic Riemannian metrics*, or *singular Riemannian metrics.* In sub-Riemannian geometry one only considers horizontal curves $c(t)$, that is, curves whose derivatives $\dot{c}(t)$ are in HOR

(when they exist). The sub-Riemannian distance between points $p, q \in Q$ is

$$d(p,q) = \inf\{ \text{ length } (c)\text{: } c \text{ a piecewise smooth horizontal curve joining } p \text{ to } q\}$$

Here length (c) is the length of the curve c with respect to the Riemannian metric k. (If there are no horizontal paths joining p to q, set $d(p,q) = \infty$.) This is independent of how k is defined in the non-horizontal directions, since c is horizontal. It is now clear that the isoholonomic problem is a special case of the

Sub-Riemannian geodesic problem. Find the horizontal curve joining p to q whose length is $d(p,q)$.

The contents of this paper is as follows. We begin by describing the configuration space of a deformable body and its geometry. Then we express the cat's problem successively as a problem in:

(A) Riemannian geometry;
(B) gauge theory;
(C) optimal control theory and sub-Riemannian geometry.

Our main result, Theorem 1 in §3, characterizes the solutions $q(t)$ to the problem of the cat with angular momentum μ as the cotangent projections to Q of solutions $(q(t), p(t))$ to a Hamilton's differential equations on T^*Q. We give a formula for the corresponding Hamiltonian function H_μ on T^*Q. H_0 is the horizontal kinetic energy.

This extends the author's previous work Montgomery [**1990**] in two ways. First, it allows for non-zero momentum. Second, it gives a formula for the Hamiltonian in terms of physical data pertinent to the problem, specifically the "locked inertia tensor".

Previously the author [**1984**] showed that this horizontal kinetic energy generates the motion of a particle under the influence of the Yang-Mills potential A. This is the content of Theorem 2 of §4, which is essentially a gauge-theoretic restatement of part of Theorem 1. Theorem 2 is contained in Montgomery [**1990**], with one error: its converse is false.

In §5 we restate the problem in the language of optimal control.

In §6 we discuss sub-Riemannian metrics and quote a theorem (Theorem 4) which enables us to prove Theorems 1 and 3 immediately.

In §7 we present an example: N point particles in space. The case $N = 3$ can be exactly solved. The differential equations are the same as those of a single charged particle under the influence of a magnetic monopole. Interesting singularities occur at the collinear configurations, where the assumption of freeness breaks down. Here the dimension of the space of zero-angular momentum vectors jumps.

Recent History. Guichardet [**1984**] observed that angular momentum define a connection. He applied his observation to molecular dynamics. See also Iwai [**1987**].

The isoholonomic problem was posed to the author by Alex Pines' in connection with some problems in nuclear magnetic resonance. The relevant bundles for Pines are the Stieffel varieties. These are the bundles of unitary k-frames over the Grassmannians of k-planes in $\mathbb{C}^{n+k}$. This problem was dealt with by the author in [**1988**].

For infinitesimal deformations of shape, Shapere and Wilczek reduced (a slight variant of) the isoholonomic problem to that of solving a second order linear o.d.e. defined on the tangent space to shape space (at the given shape). The author [**1988**] found the corresponding nonlinear o.d.e. in the case of finite deformations. This equation is Wong's equation [**1970**] for the motion of a classical spinless particle with color-charge under the influence of the Yang-Mills potential A. In the case of the planar cat G is $U(1)$ and these are the Lorentz equations for a charged particle in a magnetic field.

Recently Hamenstädt [**1986,1988**], Bär [**1988**], and the author [**1991**] made important contributions to the sub-Riemannian geodesic problem.

Future Work. Currently Zexiang Li and the author are working on applications of the ideas presented here to robot gymnastics. In future work the author plans to analyze the optimal control of the model cat of Kane and Scher [**1969**]. This model consists of two identical axially symmetric rigid bodies joined by a ball-and-socket joint at their axes.

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§2. Configuration Space and Metric for Deformable Bodies.

Let B denote a reference shape for our deformable body. For example, B might be the initial shape. The configuration space of a deformable body is a submanifold Q of the space Emb of embeddings of B into Euclidean 3-space, $\mathbb{R}^3$($\mathbb{R}^2$ if the body is planar). So,

$$Q \subset \mathrm{Map}(B, \mathbb{R}^3),$$

and a point q of Q is a map

$$q\colon B \to \mathbb{R}^3; \quad x = q(X) \in \mathbb{R}^3, \quad X \in B.$$

The group of orientation preserving rigid motions of space $SE(3)$ acts on Q by rigidly rotating and translating the body. In other words the action is given by $gq = g \circ q$, for $g \in SE(3), q \in Q$. The shape space of our deformable body is $S = Q/SE(3)$.

For example, if B consists of two rigid bodies joined together by a ball-and-socket joint, then Q is isomorphic to $\mathbb{R}^3 \times SO(3) \times SO(3)$. The components of an element $q = (g_1, g_2, \mathbf{c})$ of Q represent the orientations of the first body, and of the second body with respect to a fixed inertial frame, and the position of the joint (or, alternatively, of the center of mass). Specifically,

$$\begin{aligned} q(X) = &\mathbf{c} \text{ if } X \text{ is the joint} \\ &\mathbf{c} + g_1\mathbf{X} \text{ if } X \text{ is in body 1} \\ &\mathbf{c} + g_2\mathbf{X} \text{ if } X \text{ is in the second body} \end{aligned}$$

Shape space for this problem is $SO(3)$. The projection map $Q \to S$ is $\pi(\mathbf{c}, g_1, g_2) = {g_1}^{-1} g_2$. This matrix represents the orientation of body 2 with respect to a frame attached to body 1, that is, their *relative orientation.*

B will be endowed with a mass density $dm(X), X \in B$. This allows us to define B's total mass, the center of mass of $q(B)$, the kinetic energy of an infinitesimal deformation of $q \in Q$, etc. in the usual way. For example, the kinetic energy of an infinitesimal deformation

$$\delta q : B \to \mathbb{R}^3$$

of q is $k_q(\delta q, \delta q) = \frac{1}{2}\int < \delta q(X), \delta q(X) >_{\mathbb{R}^3} dm(X)$.

Here $\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ is the standard inner product on $\mathbb{R}^3$. The integral defines a Riemannian metric k on Q. The group $E(3)$ of rigid motions acts by isometries with respect to this metric.

We may want to ignore the translational part of our motion, since we cannot affect it by altering our shape as we fall. (Shapere and Wilczek's parmecium can affect their translation since strong friction is present.) This act of ignorance is performed by going to center of mass coordinates, that is, by setting $\int q(X)dm(X) = 0$. This defines a codimension $3, SO(3)$ invariant, totally geodesic submanifold $Q_0 \subset Q$. In symbols

$$Q_0 \cong Q/\mathbb{R}^3; \quad Q \cong \mathbb{R}^3 \times Q_0; \quad \text{Shape space} = S = Q_0/SO(3).$$

The main property of Q_0 (or Q) which we will use is that it is a Riemannian manifold on which the Lie group G acts by isometries.

§3. The Geometrical Cat.

§3.1.

We now work in the more abstract context just alluded to. We are given a Riemannian manifold Q (previously Q_0), metric k, and a Lie group G (previously $SO(3)$) which acts on Q on the left by isometries.

A vector v in T_qQ which is tangent to the group orbit $G\cdot q$ through $q \in Q$ will be called *vertical (at q)*. Vertical vectors represent infinitesimally rigid rotations of our deformable body $q(B)$. A vector v in T_qQ will be called *horizontal* if it is orthogonal to the group orbit through q. As we will soon see, (fact 1 below) v is horizontal if and only if its *angular momentum* $\mathbf{M}(q, v)$ is zero. The set of vertical vectors will be denoted V, and of horizontal vectors HOR. Thus

$$TQ = V \oplus HOR$$

the direct sum of two vector sub-bundles, assuming the rank of V is constant.

If "maximum efficiency" means minimum length, then the problem of the zero angular momentum cat is the same as the

Geometrical Cat's Problem. Find the shortest horizontal path in Q which join q_0 to gq_0. Here $q_0 \in Q$ and $g \in G$ are fixed.

This is a constrained variational problem. The function to be extremized is the length of the path. In control theory this function is called the *objective function.* It is well-known in Riemannian geometry that the same extremals are obtained if one uses the integrated kinetic energy

$$E[q] = \int k_{q(t)} \left(\frac{dq}{dt}(t), \frac{dq}{dt}(t) \right) dt \tag{3.0}$$

as the objective function, instead of the length. This is also true here. Consequently, we will use the objective function E of [3.0].

Why choose this objective function? This is an important question, to which we do not have a very satisfactory physical answer. In some circumstances it might not be the correct objective function. One justification for this choice is to suppose that frictional torques are present at the joints which are proportional to the moments of inertia of the pieces of the body. Then the integrated kinetic energy represents (energy loss) x (time), which is a desirable quantity to minimize.

More generally, we might suppose that there is some metric on shape space, perhaps empirically determined, whose integrated kinetic energy represents power expenditure. This defines a metric on HOR, so the problem still makes sense. See §6.

The Lie algebra of G will be written $\mathfrak{g}$. For q in Q, let

$$\sigma_q \colon \mathfrak{g} \to T_q Q \text{ defined by } \sigma_q(\xi) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \exp \varepsilon\xi q, \text{ for } \xi \text{ in } \mathfrak{g}.$$

denote the *infinitesimal action.* For example, in the case of our deformable body, $\mathfrak{g} = so(3) =$ skew symmetric 3×3 matrices $\cong \mathbb{R}^3$, and

$$(\sigma_q \omega)(X) = \omega \times \mathbf{q}(X), \quad X \in B, \mathbf{q} \in Q \quad \omega \in \mathbb{R}^3 \cong so(3).$$

The 3-vector ω is the angular velocity of rigid rotation.

The *momentum map* J for the G-action is the dual, $\sigma_q^* \colon T_q^* Q \to \mathfrak{g}^*$, of the infinitesimal generator σ_q, namely,

$$J \colon T^*Q \to \mathfrak{g}^* \text{ is given by } J(q,p) = \sigma_q^* p, \text{ where } p \in T_q^* Q \tag{3.1a}$$

For information and definitions regarding momentum maps, see the appendix of our paper, or Abraham and Marsden [**1978**], ch. 4.

We will use the term *angular momentum*, denoted M, for the momentum map viewed as a map from the tangent bundle TQ. We use the Riemannian metric k to identify T_qQ with T_q^*Q. Set

$$\sigma_q{}^t = \sigma_q{}^* \circ k_q \colon T_qQ \to \mathfrak{g}^*.$$

(The "t" denotes transpose.) Then

$$M(q,v) = \sigma_q^t \cdot v \text{ where } v \in T_qQ. \tag{3.1b}$$

Since M is fiberwise linear, we can view it as a one-form on Q with values in $\mathfrak{g}^*$.

We have the following basic facts concerning M.

Fact 1. $M(q,v) = \int \mathbf{q}(X) \times \mathbf{v}(X) dm(X)$, in case Q is the configuration space of a deformable body. This is the usual expression for the total angular momentum of the system. (In writing M with its values in $\mathbf{R}^3$ we identified $so(3)^*$ with $\mathbf{R}^3$ via a choice of orthonormal basis for $so(3)$.)

Fact 2. $HOR_q = (im\sigma_q)^\perp = \ker(\sigma_q{}^t) = \ker(M(q,\cdot)$.

The first equality is by definition.

Fact 3. Noether theorem: M is a constant along solutions to the Euler-Lagrange equantions for any Lagrangian $L = K - V$ on TQ with V a G-invariant function on Q.

§3.2 The Spinning Cat.

We can now formulate the

Generalized, or Spinning Cat's Problem: Find the shortest path c in Q which joins q_0 to q_1 and has constant angular momentum μ.

Our goal is to give a simple characterization of the solutions to this problem. As in Riemannian geometry, it is easier to characterize those curves which minimize length locally in the arc parameter.

Definition. A curve $c : [0,T] \to Q$ is a *local solution* to the spinning cat's problem if there exists a positive number ϵ such that for all subintervals $[a,b]$ of $[0,T]$ of length less than ϵ the restriction of c to $[a,b]$ is a solution to the spinning cat's problem with endpoints $c(a)$, $c(b)$.

Theorem 1 is a partial characterization of these local solutions. In order to to state it we must first define several functions.

The first of these functions is the "locked inertia tensor" I:

$$I_q(\xi_1, \xi_2) = k_q(\sigma_q \xi_1, \sigma_q \xi_2) \text{ for } \xi \in \mathfrak{g}.$$

for each $q \in Q$. In other words, I is the pull-back of the metric on Q to $\mathfrak{g}$. We call I *the locked inertia tensor* because I_q is the inertia tensor of the rigid body formed by locking all of the joints of the configuration q.

For each q, I_q is a symmetric nonnegative bilinear form on $\mathfrak{g}$ which is positive definite if and only if the action is *locally free* at q. (Locally free means that the isotropy group at q is discrete; equivalently $ker(\sigma) = 0$.) This is true if and only if the body $q(B)$ *is not contained within any single line*. We have the well-known formula

$$I_q = (tr\Psi_q)1 - \Psi_q$$

where

$$(\Psi_q)^{ij} = \int \mathbf{q}(X)^i \mathbf{q}(X)^j dm(X)$$

Here we have identified I_q as a symmetric 3×3 matrix by using the isomorphism between the Lie algebra $\mathfrak{g}$ and $\mathbb{R}^3$. We can also write I_q as a map from $\mathfrak{g}$ to $\mathfrak{g}^*$:

$$I_q = {\sigma_q}^t \sigma_q \tag{3.2}$$

So that, by abuse of notation, $I_q(\xi_1, \xi_2) = I_q(\xi_1)(\xi_2)$.

The other function we must define is the "optimal control Hamiltonian" for the spinning cat problem. This is the real-valued function

$$H_\mu(q,p) = K - \frac{1}{2} I^{-1}(J - \mu, J - \mu) \tag{3.3}$$

on T^*Q. In this formula K denotes the usual kinetic energy

$$K(q,p) = \frac{1}{2} k_q^{-1}(q,p).$$

k_q^{-1} and I_q^{-1} denote the inner products on T_q^*Q and $\mathfrak{g}^*$ which are induced by the inner products k_q and I_q. $I^{-1}(J - \mu, J - \mu)$ is the function $(q,p) \mapsto I_q^{-1}(J(q,p) - \mu, J(q,p) - \mu)$.

THEOREM 1. *Let $q : [0,T] \to Q$ be the cotangent projection of a solution $(q,p) : [0,T] \to Q$ to Hamilton's equations for the above Hamiltonian H_μ. And suppose that the image $q(t)(B), 0 \leq t \leq T$ of each configuration is never contained inside a single line. Then q is a local solution to the spinning cat's problem with angular momentum μ.*

REMARKS CONCERNING THE ZERO-ANGULAR MOMENTUM CASE.: (1) J is a conserved quantity for the $\mu = 0$ Hamiltonian: $\{H_0, J\} = 0$. The value of this constant $J(q(t), p(t))$ can be any element of $\mathfrak{g}^*$ even though *every* solution $(q(t), p(t))$ of the corresponding $\mu = 0$ Hamiltonian system satisfies $M(q(t), \dot{q}(t)) = 0$.

(2) Fix the value of this constant: $J = \alpha = const$. Then we can view the optimal control flow as the motion of a particle in the field of the "effective potential" defined by the second term of the Hamiltonian, $-\frac{1}{2}I_q^{-1}(\alpha, \alpha)$.

(3) H_0 is the *horizontal kinetic energy* as defined by orthogonal direct sum decomposition $T^*Q = V^* \oplus HOR^*$. In other words, if $\mathsf{P}_{HOR} : T_q^*Q \to HOR_q^*$ denotes the corresponding projection then

$$H_0(q,p) = \frac{1}{2}k_q{}^{-1}\left(\mathsf{P}_{HOR^*}(q) \cdot p, \mathsf{P}_{HOR^*}(q) \cdot p\right) \tag{3.4}$$

This can be seen by noting that the effective potential of the previous remark is the vertical kinetic energy.

(4) §7.2.3 discusses an example where the deformable body is allowed to become collinear so that the hypothesis of Theorem 1 fails. The optimal curves are then concatenations of solutions to these Hamilton's equations with the concatenation points occuring at the collinearities. The corresponding curves have derivative discontinuities at these points.

§4. The Gauge-theoretic Cat.

As long as our deformable body is never contained within a single line in 3-space the G action is *free*. This, together with the compactness of G, implies that $Q \to S = Q/G$ forms a principal G-bundle and that HOR defines a connection on this principal bundle.

Guichardet [**1984**], and later Shapere and Wilczek [**1987,1989**] give a formula for the corresponding connection one-form (gauge field) A on Q:

$$A_q = I_q^{-1} M(q, \cdot\,)\colon T_q Q \to \mathfrak{g}. \tag{4.1a}$$

Equivalently (see equations 3.1 and 3.2)

$$A = (\sigma^t \sigma)^{-1} \sigma^t. \tag{4.1b}$$

It is immediate from fact 2 of §3 that A satisfies the desired property:

$$\ker A = HOR = \text{ set of deformations with zero angular momentum}$$

It also satisfies

$$\mu \cdot A_q \in J^{-1}(\mu) \subset T^*Q$$

for every $\mu \in \mathfrak{g}^*$ (Any connection satisfies this last property.) The connection A is called nowadays the "natural mechanical connection" or "master gauge". In words, "the natural mechanical connection on shape space is the inverse inertia tensor times the angular momentum".

The physical meaning of this connection is elucidated by considering the following procedure. Let $s(t)$ be a path in shape space and $q(0) \in Q$ be an initial configuration of the body. What is the full motion $q(t)$ of the body as it deforms through space? Suppose we also know that the initial total angular momentum M of the body is zero. It remains zero by conservation of angular momentum. Thus $q(t)$ is a horizontal path which projects onto $s(t)$. It follows that $q(t)$ is recovered from $s(t)$ by *parallel translating* $q(0)$ along $s(t)$ with respect to the connection A.

The curve q described above is called the horizontal lift of s. If $s = \pi \circ q$ is a closed curve then there is a unique group element g such that $q(1) = gq(0)$. This element is called the *holonomy* of s (based at $q(0)$).

The metric k on Q induces a metric k_S on shape space by declaring the restriction of $d\pi_q\colon T_q Q \to T_s S$ to HOR_q to be an isometry. This makes $Q \to S$ into a *Riemannian submersion*.

We can now rephrase the geometric (= zero angular momentum) cat's problem as the **isoholonomic problem: Among all loops in shape space based at an initial shape s_0 find the shortest one whose holonomy is g.** More generally we have the **isoparallel problem: Fix an initial and final shape s_0 and s_1 and also a G-equivariant map $g : \pi^{-1}(s_0) \to \pi^{-1}(s_0)$. Among all curves in shape space which join the given shapes s_0, s_1 find the shortest one whose parallel transport operator is g.**

In **[1990]** I computed the formal Euler-Lagrange equations for this problem and showed that it reduces to the differential equation which governs the motion of a particle traveling through the Riemannian manifold S while under the influence of the gauge field A. These are the so-called "Wong equations" [**Wong**] or "Kerner equations" [**Kerner**]. They are equations for a curve $e(t)$ in the co-adjoint bundle $\mathfrak{g}^*(Q)$ which is a vector bundle over S with typical fiber $\mathfrak{g}^*$, the dual of the Lie algebra of our group G. This bundle is an associated bundle to Q and can be defined by the formula $\mathfrak{g}^*(Q) = Q \times_{Ad^*} \mathfrak{g}^*$. It is naturally isomorphic to V^*/G where V^* denotes the dual of the vertical bundle $V = \ker d\pi$ over Q.

To describe the equations, write $s(t) = \pi(e(t)) \in S$ and $\dot{s} = \frac{ds}{dt} \in TS$. (We occasionally abuse notation and denote any projection by "π".) Let D denote the connection on $g^*(Q)$ induced by the connection A on Q. In coordinates, $De = de + ad^*(A)e$. Let ∇ be the Riemannian (Levi-Civita) connection on S induced by the metric k_S. Let F denotes the curvature of A which we can think of as a two-form on S with values in the adjoint bundle $g(Q) = Q \times_{Ad} \mathfrak{g} \cong V/G$. Then $e \cdot F(\dot{\gamma}, \cdot)$ is a one-form along s (a force) and $k_S \cdot e \cdot F(\dot{\gamma},)$ is a vector field along s. Wong's equations are

$$\nabla_{\dot{s}}\dot{s} = k_S \cdot e \cdot F(\dot{s}, \cdot) \tag{4.3a}$$

$$\frac{De}{dt} = 0. \tag{4.3b}$$

Wong's equations are second order in s and first order in the fiber. They can be written in first order form by adding the equation

$$\dot{s} = k_S(s)^{-1} y, \tag{4.3c}$$

where $y \in T^*_s S$, the cotangent bundle to shape space. Now use this to rewrite the previous differential equations in terms of y and $\dot{y}$. The result is a system of first order differential equations for a curve (s, y, e) in the vector bundle $g^*(Q) \oplus T^*S$ over S. We call this vector bundle the "phase space of a (co-adjoint) particle in a Yang-Mills field".

Definition: A curve c in S is a motion of a Yang-Mills particle if it is the projection to S of some solution of the above system of differential equations in $g^*(Q) \oplus T^*S$.

The above equations for a particle in a Yang-Mills field can be written in Hamiltonian form. See Montgomery [**1984**] and references to Sternberg and Weinstein therein. In fact, these equations are obtained by reducing the equations on T^*Q defined by the optimal control Hamiltonian (see Theorem 1) by the action of the group G. To see this, use the connection A to define a G equivariant isomorphism:

$$T^*Q = V^* \oplus HOR^* \cong \mathfrak{g}^* \times \pi^*T^*S$$

This is the dual of the usual vertical-horizontal splitting of TQ. Upon dividing by G we obtain an isomorphism

$$(T^*Q)/G \cong \mathfrak{g}^*(Q) \oplus T^*S. \tag{4.4}$$

Since H_0 is a G-invariant function on T^*Q its Hamiltonian vector field pushes down to define a Hamiltonian vector field on the reduced (Poisson) manifold $g^*(Q) \oplus T^*S$, which is the phase space for a particle in a Yang-Mills field.

There is an alternative, older, viewpoint on the motion of a Yang-Mills particle which is due to Kaluza-Klein. Let β be a fixed adjoint-invariant positive-definite inner product on $\mathfrak{g}$. So, if G is semi-simple β is a multiple of the Killing form and in particular for the case of most interest to us, $G = SO(3)$, β is the standard inner product on $\mathbb{R}^3$. Define a new metric k_{biv} on Q, the "bi-invariant Kaluza-Klein metric", by

$$\begin{aligned} k_{biv}(v_1, v_2) &= k(v_1, v_2) \text{ if } v_1 \text{ and } v_2 \text{ are horizontal} \\ &= \beta(\xi_1, \xi_2) \text{ if } v_1 \text{ and } v_2 \text{ are vertical with } v_i = \sigma\xi_i \\ &= 0 \text{ if } v_1 \text{ is vertical and } v_2 \text{ is horizontal} \end{aligned} \tag{4.5}$$

Definition. A Kaluza-Klein geodesic is a geodesic on Q with respect to this metric.

Let $H_{biv} = \frac{1}{2}k_{biv}{}^{-1}(q)(p,p)$ denote the corresponding kinetic energy , a function on T^*Q. One calculates (see Montgomery [**1984**]) that

$$H_{biv} = H_0 + \frac{1}{2}\beta^{-1}(J, J)$$

Also the Poisson bracket of the second term, $\beta^{-1}(J, J)$ with *any* G-invariant function is zero. Consequently the push-down (to the Yang-Mills phase space) of the Hamiltonian vector field of this second term is zero. It follows that the push-downs for H_0 and H_{biv} are equal. This proves

THEOREM 2. *The following statements regarding a curve c in S are equivalent (1) c is the motion of a particle in a Yang-Mills field. (2) c is the project to S of a solution to Hamilton's equations for the optimal control Hamiltionian H_0 (3) c is the projection to S of a Kaluza-Klein geodesic on Q.*

Theorem 2 was proved in Montgomery [**1984**]. As an immediate corollary to Theorems 1 and 2 we have

THEOREM 3. *(1) If c is a motion of a particle in a Yang-Mills field then it is a local solution to the zero-angular momentum cat's problem. (2) If q is a Kaluza-Klein geodesic then its projection $c = \pi \circ q$ to S is a local solution to the zero angular momentum cat's problem. The corresponding zero-angular momentum curve $\tilde{q}(t)$ in Q is obtained by taking the horizontal lift of the projection c.*

Theorem 3 can be found in Montgomery [**1990**]. There you can also find the following formula for the passage from q to $\tilde{q}$:

$$\tilde{q}(t) = \exp(-t\xi)q(t)$$

where $\xi = A_q \cdot \dot{q}$, a time-independent Lie algebra element.

§5. Optimal Control.

We can view tangent vectors u to shape space as *control variables.* Let

$$h_q \colon T_s S \to HOR_q \subset T_q Q, \text{where } s = \pi(q),$$

denote the operation of horizontal lift. It can be defined as the unique linear operator from $T_s S$ to $T_q Q$ whose image is HOR_q and which is a right inverse to the differential of the projection: $d\pi_q \circ h_q$ = identity on $T_s S$. Set

$$\begin{aligned} X_\mu(q) &= k_q{}^{-1}(\mu \cdot A_q) \\ &= \sigma_q(I_q^{-1}(\mu)), \end{aligned}$$

a vector field on Q. According to equation [4.2], $M(q, X_\mu(q)) = \mu$. In fact, $X_\mu(q)$ is the shortest vector in the affine subspace $\{v : M(q,v) = \mu\}$ of $T_q Q$.

Any tangent vector $\dot{q}$ to Q at q which satisfies $\mathbf{M}(q,\dot{q}) = \mu$ can be expressed uniquely in the form $\dot{q} = h_q u + X_\mu(q)$. And in this case $k(\dot{q},\dot{q}) = k_S(u,u) + I_q^{-1}(\mu,\mu)$. It follows that the spinning cat problem is equivalent to the following problem in optimal control :

$$\text{given } \dot{q} = h_q u + X_\mu(q) \tag{5.1}$$

with

$$q(0) = q_0, q(1) = q_1 \tag{5.2}$$

minimize

$$\frac{1}{2} \int k_S(u(t), u(t)) + V_\mu(q) dt. \tag{5.3}$$

where

$$V_\mu(q) = I_q^{-1}(\mu, \mu)$$

This is the standard form of an optimal control problem. 5.1 is called the control law. 5.2 says that the control steers q_0 to q_1. 5.3 is called the cost function or value function. X_μ is called the drift vector field.

This reformulation is important for two reasons. First, it allows us to view the problem as a feedback control problem. This is probably the correct point of view for of the cat; when blindfolded she usually fails to land on her feet (Kane [**1989**], private conversation). Second, it makes sense even if the metric k_S on shape space has no relation the metric on Q. For example , k_S might be an empirically or analytically determined power dissipation law. This is the case for Shapere and Wilczek's microorganisms [**1987**]. And V_μ might be some "potential" which one must keep small.

§6. Sub-Riemannian Geometry.

§6.1 Basics of Sub-Riemannian Geometry.

The geometrical cat's problem of §3 is a special case of the problem of finding sub-Riemannian geodesics. This more general point of view provides a straightforward proof of our Theorem 1. In fact, we simply quote a result of Rayner [**1967**] or Hammenstadt [**1986**] which we have summarized here as Theorem 4.

Definition A sub-Riemannian metric on the manifold Q consists of a (typically nonintegrable!) distribution $HOR \subset TQ$, together with a smoothly varying positive definite product $\kappa(q)$ on HOR_q.

A contravariant object, for example a curve or vector, is called "horizontal" if it is tangent to the given distribution. In general, we only consider horizontal objects.The *length* of the horizontal curve γ is

$$\text{length } [\gamma] = \int \sqrt{\kappa(\dot\gamma(t), \dot\gamma(t)} dt \tag{6.1}$$

The **sub-Riemannian geodesic problem** is the problem of finding the shortest horizontal curve joining two fixed endpoints $q_0, q_1 \in Q$.

Definition: A *minimizing sub-Riemannian geodesic* is a rectifiable horizontal curve which is the shortest such curve among all such curves joining its endpoints.

Definition: A locally minimizing geodesic is a rectifiable horizontal curve for which each sufficiently small subarc of is a minimizing sub-Riemannian geodesic. (Compare with the definition of local solution to the spinning cat problem.)

If we take HOR to be as in the previous sections and take κ to be k restricted to the horizontal distribution then the zero-angular momentum cat problem is precisely the problem of finding minimizing sub-Riemannian geodesics.

Remark Sub-Riemannian metrics are also referred to as *Carnot-Caratheodory metrics, non-holonomic Riemannian metrics, or singular Riemannian metrics.*

A sub-Riemannian structure defines, and is defined by, a constant rank " co-metric" g. This is a symmetric contravariant two-tensor whose rank is the dimension of the distribution. We can think of it as a symmetric vector bundle endomorphism $g\colon T^*Q \to TQ$ in which case it is defined by the requirements:

(1) image(g) $= HOR$

(2) if $v = g(q)(p) \in T_qQ$, then $k_q(v,v) = p(g(q)(p))$.

We can also think of $g(q)$ as a bilinear form on T_q^*Q in which case we write:

$$g(q)(p_1, p_2) = p_1(g(q)(p_2)).$$

The fiber-quadratic form

$$H_0(q,p) = \frac{1}{2} g(q)(p,p) \tag{6.2}$$

is called the ***horizontal kinetic energy***, ***sub-Riemannian kinetic energy*** or ***optimal control Hamiltonian***.

THEOREM 4. *[Rayner [**1967**], Hammenstädt [**1990**]] Let $(q(t), p(t))$ be a solution to Hamilton's equations with the Hamiltonian function H_0 of equation [6.2]. In other words suppose $(q(t), p(t))$ satisfies the differential equations*

$$\frac{dq^i}{dt} = \Sigma g^{ij} p_j; \quad \frac{dp_i}{dt} = -\frac{1}{2}\Sigma \left[\frac{\partial(g^{kj})}{\partial q^i}\right] p_k p_j,$$

where (q^i, p_i) are canonical coordinates on T^*Q, and $H_0 = \frac{1}{2}\Sigma g^{kj}(q) p_k p_j$. Then $q(t)$, the cotangent projection of this curve, is a locally minimizing sub-Riemannian geodesic.

This theorem seems to have first been proved by Rayner [**1967**]. It was later proved and strengthened by Hammenstädt. See her [**1990**] paper. We will not reprove this theorem. Instead, we will content ourselves by in §6.3 with calculating the Hamiltonian H_0 (and H_μ) and by showing that in the case of a deformable body that the H_0 of Theorem 4 is equal to the H_0 of Theorem 1. Thus Theorem 1 is a restatement of Theorem 4.

The converse to Theorem 4 is false. Unfortunately, the converse has been stated as a theorem in many papers going back at least as far as Rayner! Bär in his thesis gives examples of a locally minimizing horizontal curve which does not satisfy the geodesic equations. (Sufficiently small subarcs of Bär's curve admits a "cotangent lift" which satisfies Hamilton's equations but these cannot be smoothly spliced together.) Montgomery [**1991**] gives examples of globally minimizing sub-Riemannian geodesics which do not satisfy the geodesic equations. In each of these examples the distribution generates the tangent bundle to Q. That is, it satisfies the conditions of Chow (also called the conditions of Hörmander ; see the next section.) Such "pathologies" do not occur in Riemannian geometry.

Some History: Sub-Riemannian geometry appears in the study of CR and contact structures, hypoelliptic operators, the analysis of rigidity problems for spaces with nonpositive curvature such as complex or quaternionic hyperbolic space, and the collapsing of Riemannian manifolds. There is a fair-sized literature on sub-Riemannian geometry. Among the works that have come to our attention are Hermannn [**1962,1973**] Rayner[**1967**], Hamenstädt [**1986,1988,1990**], Bär[**1989**], Brockett [**1981,1983**], Baillieul [**1975**], Gunther [**1982**], Strichartz [**1983,1989**], and Taylor [**1989**]. Vershik and V. Ya Gershkovich [**1988**] give a kind of review with a summary of facts and some intriguing pictures. The sub-Riemannian geodesic problem is a special case of the problem of Lagrange in the Calculus 0f Variations. This is treated in generality by Carathe'odory [final chapter] and Bliss [**1930**].

§6.2. Chow, Ambrose-Singer and Controllability.

There may be no horizontal paths which join the point q_0 to the point q_1. In this case there are, of course, no solutions to the corresponding sub-Riemannian geodesic problem. To avoid this situation we would like conditions on a given distribution which insure that any pair of points $(q_0, q_1) \in Q \times Q$ can be joined by a horizontal path. A distribution, or more generally, a control law, which satisfies such a property is called *controllable*.

Let $E_i, i = 1, 2, \ldots$ be a local frame for the horizontal distribution. Form the iterated Lie brackets $[E_i, E_j], [E_i, [E_j, E_k], \ldots$ and evaluate these at $q \in Q$.

Definition: We say that Chow's condition holds at q if these vectors, together with the $E_i(q)$, eventually (i.e. after enough iterates of Lie brackets are taken) span all of T_qQ.

THEOREM. *[Chow, Rashevsky] If Q is connected and Chow's condition holds everywhere, then any two points of Q can be joined by a smooth horizontal curve.*

Chow's original theorem is actually slightly stronger than this. Chow's work actually contains no Lie brackets; instead, his original condition is phrased in terms of push-forwards of vector fields and is a localization of the above theorem about a given horizontal curve.

Remark. What we are calling Chow's condition is very often referred to as "Hörmander's condition".

A corollary to Chow's theorem is the Ambrose-Singer [**1953**] Theorem for connections on principal bundles. (See also Kobayashi-Nomizu [**1963**, p. 83-89]). Suppose we are in the previous setting in which $Q \to S$ is a principal G-bundle. Let $X, Y, Z, \ldots$ be horizontal vector field on Q. Recall that the *curvature* F of the connection A at can be defined by the equality: vertical part of $([X, Y](q)) = \sigma_q(F_q(X, Y))$. Similarly, the vertical part of $[Z, [X, Y]](q)$ is equal to $\sigma_q(D_Z F_q(X, Y))$ and analogous statements hold for the higher covariant derivatives of the curvature. Applying Chow's theorem, we obtain the following weak version of the Ambrose-Singer Theorem.

THEOREM. *[Ambrose-Singer] Suppose Q is connected and let q be some point of Q. Let $\Delta(q)$ denote the Lie subalgebra of the Lie algebra of G which is generated by the values of the curvature $F(X, Y)$ at q, together with all of its covariant derivatives $D_Z F(X, Y), D_W D_Z F(X, Y), \ldots$ evaluated at q as $X, Y, Z, \ldots$ range through HOR_q. If $\Delta(q)$ is the entire Lie algebra then any two points of Q can be joined by a smooth horizontal path.*

§6.3 Calculation of the Hamiltonian.

We will calculate the Hamiltonian H_μ from the constrained Lagrangian $L : HOR \to \mathbb{R}$, $L(q, u) = \frac{1}{2}\kappa_q(u, u) + V_\mu(q)$. This does not constitute a proof of any of the theorems, rather it lends to their credibility. (We have proved the theorems merely by quoting the theorems of others.) Our method of calculation is identical to the method used for prescribing the Hamiltonian of the Pontrjagin maximum principle. This in turn is identical to the method of Legendre transform used in classical mechanics once we realize that $\dot{q} = u + X_\mu(q)$ where $u \in HOR_q$. Note also that $L(q, u) = \frac{1}{2}k_q(\dot{q}, \dot{q})$. See §5 where we note that $V_\mu(q) = \frac{1}{2}I_q^{-1}(\mu, \mu)$ for the spinning cat.

Define

$$\tilde{H}(q, p, u) = p(u + X_\mu(q)) - L(q, u)$$

where $q \in Q$, $p \in T_q^*Q$,$u \in HOR_q$ and X_μ is the "drift vector field" introduced at the beginning of section 5. If $\mu = 0$ then $X_\mu = 0$ which is the case stated in Theorem 1. The Hamiltonian H is by definition is the Legendre transform of L:

$$H(q,p) = inf_u \tilde{H}(q,p,u)$$

By elementary calculus this infimum is realized by the unique vector $u \in HOR_q$ such that $p = \kappa_q(u,\cdot)$ when restricted to HOR_q. We can reexpress this relationship by

$$u = g(q)(p)$$

Evaluate $\tilde{H}(q,p,g(q)(p))$ to find that

$$H(q,p) = \frac{1}{2}g(q)(p,p) + p(X_\mu(q)) - V_\mu(q) \tag{6.3}$$

which is the desired result.

In the case of our deformable body κ is the restriction of the Riemannian metric k on Q and there is a simple formula for the cometric g. Let

$$\mathsf{P}_{HOR}: TQ \to HOR; \mathsf{P}_V : TQ \to V$$

denote the k-orthogonal projections. Let $k^{-1}: T^*Q \to TQ$ be the k-induced isomorphism. Then

$$g = \mathsf{P}_{HOR} \circ k^{-1} = k^{-1} - \mathsf{P}_V \circ k^{-1}.$$

Moreover

$$\mathsf{P}_V = \sigma \circ A$$

where A is the natural mechanical connection of §4. Combining these formulas with the formula for A and the definition of $\mathbf{M}$, we find that the H_0 of Theorem 1 (equations [3.3 or .4]) equals the H_0 of Theorem 4 (equation [6.2]). Similarly the H above is the H_μ of Theorem 1.

§7. Examples. Point masses.

§7.1 N Point Masses.

The body B consists of N point masses in d-dimensional Euclidean space. We are most interested in the case $d = 3$ but it is no extra work to follow the general case, at least for awhile. The masses are $m_1, \ldots, m_N$ and have positions $x_1, \ldots, x_N \in \mathbf{R}^d$. We assume that we are in the center of mass frame: $\Sigma m_a x_a = 0$. Then Q is a $d(N-1)$-dimensional subspace of $\mathbf{R}^{dN}$. Notice that we do not delete the collision configurations $\{x_a = x_b\}$. To imagine changing the shape of a configuration when $d = 3$ suppose that each mass slides back and forth upon a massless rod and that these rods can be swung about by means of massless joints attached to the center of mass.

An element of the shape space $S = Q/SO(d)$ can be visualized by connecting the N points in order by line segments so as to form an N-gon. For example, if $N = 3$ then S is the space of triangles. Coordinatizing S for large N is an interesting problem in classical invariant theory (H. Weyl [**1938**])

The metric on Q is $\Sigma m_a \|dx_a\|^2$. $\mathbf{M} = \Sigma m_a x_a \wedge v_a$ is the angular momentum. When $d = 3$ we have $\wedge = \times$, the vector cross product. The inertia tensor $I = \Sigma m_a {x_a}^i {x_a}^j$. Thus $I(\xi, \xi) = \Sigma m_a {x_a}^i {x_a}^j \xi_i^k \xi_j^k$. (For $d = 3$ the isomorphism of $\mathfrak{so}(3)$ with $\mathbf{R}^3$ converts this into the earlier formula for I.) I is nondegenerate as long as the positions x_a of the masses span either a subspace of dimension $d-1$ or all of $\mathbf{R}^d$. We will say that such configurations are in general position. For instance when $d = 3$ a configuration is in general position provided all of its points do not lie on the same line.

The action of $SO(d)$ is free on the set of points in general position and so $Q \to S$ forms a principal $SO(d)$ bundle upon restriction to the configurations in general position. As in §4 the distribution $\{\mathbf{M} = 0\}$ of zero-angular momentum deformations defines a connection on this principal bundle. Guichardet [**1984**] showed that this connection satisfies the conditions of the Ambrose-Singer Theorem (§6.2) provided $N > d$. Consequently any two generic configurations can be connected by a zero angular momentum path when $N > d$. The condition "can be joined by a zero-angular momentum path" is a closed condition on the set of pairs of points in Q. It follows that any two configurations can be joined by a zero angular momentum path when $N > d$.

When $N = d$ Guichardet has shown that a generic configuration moving with zero-angular momentum must remain on the $d-1$ dimensional subspace which it initially spans *as long as it remains generic.* (Exercise: use Cartan's lemma for two-forms to prove this.) Note that the fact that the center of mass is zero says that the d vectors cannot be linearly independent.

When $N = d$ it is still possible to join any two configurations by a zero-angular momentum path. By the above observation the only way to do this when the two configurations span different hyperplanes in $\mathbf{R}^d$ is to pass through some intermediate degenerate configuration, that is, a configuration whose span has codimension 2 or greater. These degenerate configurations thus act as "switching yards" between the different hyperplanes. When $N = d = 3$ we will show explicitly how to connect triangular configurations spanning different planes by passing through collinear configurations.

The optimal control Hamiltonian for our zero angular momentum "cat" problem here is given by the formula of Theorem 1:

$$H_0 = K - \frac{1}{2} I^{-1}(J, J)$$

Here $K(x,p) = \frac{1}{2}\Sigma\|p_a\|^2/m_a$ is the kinetic energy and $J = \Sigma x_a \wedge p_a$ is the angular momentum written in terms of momentum variables.

An interesting fact about this Hamiltonian is that it is *collective* for the linear symplectic group $Sp(d)$ of $\mathbf{R}^{2d}$. To see this, note that we can write $T^*Q \subset \mathbf{R}^{2d} \otimes \mathbf{R}^N$, the tensor product of the symplectic vector space $\mathbf{R}^2 d$ with the inner product space $\mathbf{R}^N$ where the inner product is defined by the masses. In general, if (Z, ω) and $(E, \langle , \rangle)$ are two such vector spaces then the symplectic form on $Z \otimes W$ is given by $(z_1 \otimes e_1), (z_2 \otimes e_2) \mapsto \omega(z_1, z_2)\langle e_1, e_2 \rangle$. The groups $Sp(Z)$ and $O(E)$ act in a linear symplectic fashion on $Z \otimes E$ and form one of the standard examples of a Howe dual pair. The $Sp(Z)$ momentum map is given by $\Phi(\Sigma z_\mu \otimes e_\mu) = \Sigma z_\mu \langle e_\mu, e_\nu \rangle z_\nu \in \mathfrak{sp}(Z)^* = S^2(Z)$. Here we have identified the Lie algebra $\mathfrak{sp}(Z)$ of $Sp(Z)$ with the space $S^2(V^*)$ of homogeneous quadratic polynomials on Z by taking such a polynomial to its corresponding linear Hamiltonian vector field. In our case, let $x^i, v^j \in \mathbf{R}^{2d}$ be symplectic coordinates so that $\omega = \Sigma dx^i \wedge dv^i$. The metric on $\mathbf{R}^N$ is given by $\Sigma m_a (dt_a)^2$. Then we have nonsymplectic coordinates x^i_a, v^j_b on $\mathbf{R}^{2d} \otimes \mathbf{R}^N$. The $Sp(d)$ momentum map Φ takes values in the space of $2d \times 2d$ symmetric matrices. It consists of the four $d \times d$ blocks $\Sigma m_a x_a^i x_a^j$, $\Sigma m_a x_a^i v_a^j, \Sigma m_a v_a^i x_a^j, \Sigma m_a v_a^i v_a^j$. By inspection H_0 can be written as a smooth function of the entries of Φ. Thus, by definition, H_0 is collective for the group $Sp(d)$. This means that by solving *one* Hamiltonian differential equation on $\mathfrak{sp}(d)^*$ we will have solved our cat's problem *for all* N. Unfortunately, I do not see how to solve this universal system, even for $d = 2$ or 3.

§7.2. N = 3 Point Masses in the Plane: Exact Solutions.

We will use complex coordinates so that $\mathbb{R}^2 = \mathbb{C}$. Up to equation [7.2.10] our formulas can be found in Iwai [**1987a**].

The configuration space is

$$Q = \{m_1 z_1 + m_2 z_2 + m_3 z_3 = 0\} \subset \mathbb{C}^3.$$

By Graham-Schmidt (see Iwai)

$$q_1 = \sqrt{\frac{m_1 m_3}{m_1 + m_3}}(z_1 - z_3) \tag{7.2.1a}$$

$$q_2 = \sqrt{\frac{m_2(m_1 + m_3)}{m_1 + m_2 + m_3}}\left(z_1 - \frac{m_1 z_1 + m_3 z_3}{m_1 + m_3}\right) \tag{7.2.1b}$$

define orthonormal coordinates on Q, namely,

$$k = |dq_1|^2 + |dq_2|^2. \tag{7.2.2}$$

The rotation group acts according to

$$e^{i\Psi}(q_1, q_2) = (e^{i\Psi}q_1, e^{i\Psi}q_2).$$

and this action is free on $Q \setminus \{(0,0,0)\} = \mathbb{C}^2 \setminus \{0\}$. The natural mechanical connection on $Q \setminus \{0\}$ is

$$A = \frac{1}{r}\Im(\bar{q}_1 dq_1 + \bar{q}_2 dq_2), \tag{7.2.3a}$$

where

$$r = |q_1|^2 + |q_2|^2 = I_q \tag{7.2.3b}$$

is the moment of inertia of the configuration (q_1, q_2). (Check that $A = 1$ on the infinisimal generator (iq_1, iq_2) of the action and that A annihilates the perpindicular to the generator.) The projection

$$(7.2a) \qquad \pi\colon Q \to S, \text{ restricted to } Q \setminus \{0\},$$

is equal to the radial extension

$$(7.2.4b) \qquad \pi\colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$$

of the well known Hopf fibration $S^3 \to S^2$. Specifically, we can identify S topologically with $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. Then

$$(7.2.5) \qquad \pi((q_1, q_2)) = (2\bar{q}_1 q_2, |q_1|^2 - |q_2|^2) = (x + iy, z).$$

The origin $0 = \pi((0,0,0))$ is a distinguished nonsmooth point of S (an orbifold point). It represents the special shape consisting of all three masses sitting at the origin.

We have $r = \sqrt{x^2 + y^2 + z^2} = |q_1|^2 + |q_2|^2$. (Compare [7.2.3b].) It follows that π maps the 3-sphere $S^3(\sqrt{r})$ of radius $\sqrt{r}$ centered at the origin in $Q = \mathbb{C}^2$ to the sphere $S^2(r)$ of radius r in $S = \mathbb{R}^3$. Setting $r = 1$, we obtain the standard Hopf fibration $S^3 \to S^2$.

Warning: the induced (k_S) metric on the sphere $S^2(r)$ is that of a sphere of radius $\frac{1}{2}\sqrt{r}$. See formula [7.2.8].

We have the following facts concerning the suspended Hopf fibration [7.2.4b]. The curvature F of the connection A is that of a point magnetic monopole with strength $\frac{1}{2}$ at the origin 0 of $S = \mathbb{R}^3$. In symbols,

$$(7.2.7a) \qquad F = \frac{1}{2} d\Omega,$$

where $d\Omega$ is the usual solid angle two-form

$$d\Omega = \frac{x dy \wedge dz + z dx \wedge dy + y dz \wedge dx}{r^3}$$

If we identify two-forms F and vector fields $\mathbf{F}$ on $\mathbb{R}^3$ in the usual way: $F(v_1, v_2) = < \mathbf{F}, v_1 \times v_2 >$ then

$$(7.2.7b) \qquad \mathbf{F}(\mathbf{x}) = \frac{1}{2}\mathbf{x}/r, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$$

Warning: The d of $d\Omega$ is conventional; it does not mean the exterior derivative. $d\Omega$ is not an exact differential but it is closed.

The induced metric on $S \setminus \{0\}$ is calculated to be

$$k_S = \frac{1}{4r}(dx^2 + dy^2 + dz^2) \tag{7.2.8}$$

where $dx^2 + dy^2 + dz^2$ is the standard metric on $\mathbf{R}^3$. In particular it is conformal to the standard metric. It follows from [7.2.7b] that the Lorentz force

$$k_S^{-1} eF(\dot{\mathbf{x}}, \cdot) = \lambda \mathbf{x} \times \dot{\mathbf{x}}, \tag{7.2.9}$$

where λ is a scalar. In fact $\lambda = -2e$.

Formula [7.2.7] for the curvature is well-known. Formula [7.2.8] for the shape metric is not as well-known but can be found in Iwai [**1987a**]. For completeness we will derive these formulas. They are somewhat simpler if we use polar coordinates on $\mathbf{C}^2$:

$$(q_1, q_2) = \sqrt{r}(\cos(\theta/2)e^{i\Psi},\ \sin(\theta/2)e^{i\phi}e^{i\Psi}). \tag{7.2.10}$$

with $\theta/2, \varphi$, and Ψ being angles. In these coordinates the $SO(2)$ action is $\Psi \to \Psi + \Delta\Psi$. One finds that

$$\pi(q_1, q_2) = r(\sin\theta e^{i\varphi},\ \cos\theta).$$

It follows that (r, θ, φ) are the standard polar coordinates on $\mathbf{R}^3$ with θ the angle which (x, y, z) makes with the z-axis. Differentiating [7.2.9] and plugging this in to [7.2.2, .3] yields

$$A = d\Psi + \sin^2(\theta/2)d\varphi \tag{7.2.11}$$

and

$$k = \frac{1}{4r}\{dr^2 + r^2 d\theta^2\} + r\{\sin^2(\theta/2) - \sin^4(\theta/2)\}d\varphi^2 + rA^2.$$

The metric k_S is obtained by setting $A = 0$ in this last equation. Since $\sin^2(\theta/2) - \sin^4(\theta/2) = [\frac{1}{2}\sin\theta]^2$ we obtain

$$\begin{aligned} k_S &= \frac{1}{4r}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \\ &= \frac{1}{4r}[dx^2 + dy^2 + dz^2] \end{aligned}$$

which is formula [7.2.8]. To obtain [7.2.7a] take the exterior derivative of [7.2.11] and note that $\sin^2(\theta/2) = \frac{1}{2}(1 - \cos\theta)$. This yields

$$F = dA = \frac{1}{2}\sin\theta(d\theta \wedge d\varphi)$$

which is the expression for $\frac{1}{2}d\Omega$ in polar coordinates.

The Lorentz equations on $\mathbf{S} \setminus \{0\}$

Our A, being an $SO(2)$ gauge field, is just a one-form (vector potential). It follows that the optimal control equations, which are the equations of motion for a particle in a Yang-Mills field (§4, Theorem 3), are just the Lorentz equations of a particle in the magnetic field F except that the metric on S is not the standard metric on $\mathbf{R}^3$ but rather the funny metric k_S. These Lorentz equations are

$$\nabla_{\dot{\mathbf{x}}}\dot{\mathbf{x}} = \lambda \mathbf{x} \times \dot{\mathbf{x}}$$

where

$$\lambda = \text{ constant } = -2e. \tag{7.2.12b}$$

and where ∇ is the covariant derivative for the shape metric k_S.

Symmetry and Momentum Map. The rotation group $SO(3)$ acts on $\mathbf{R} \times T^*S \cong \mathbf{R} \times \mathbf{R}^3 \times \mathbf{R}^3$ by $R(e, \mathbf{x}, \mathbf{v}) = (e, R\mathbf{x}, R\mathbf{v})$. This action is a Hamiltonian action and preserves the Wong Hamiltonian

$$H(e, \mathbf{x}, \mathbf{v}) = 4r\|\mathbf{v}\|^2.$$

By virtue of these facts, the momentum map

$$\mathbf{L}\colon \mathbf{R} \times \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3 \cong so(3)^*$$

for the $SO(3)$ action, which is given by

$$\mathbf{L}(e, \mathbf{x}, \mathbf{v}) = \mathbf{x} \times \mathbf{v} + \frac{1}{2}e\mathbf{x}/r \tag{7.2.13}$$

is constant along Wong trajectories. This formula for the momentum map is well- known. See for example Jackiw and Manton [**1980**]. For completeness we present a derivation of it in the appendix.

Solutions. Let $(e, \mathbf{x}(t), \mathbf{v}(t))$ be a solution to our Wong's equations, and $\mathbf{L}_0$ the value of the momentum map on this solution. Then

$$\mathbf{x} \cdot \mathbf{L}_0 = \mathbf{x} \cdot \mathbf{L}(e, \mathbf{x}, \mathbf{v}) = \frac{e}{2} r. \tag{7.2.14}$$

This is the equation of a cone $C = C(\mathbf{L}_0, e)$ in $\mathbf{R}^3$ since $\mathbf{L}_0$ is a constant vector in $\mathbf{R}^3$. The curve $\mathbf{x}(t)$ is constrained to lie on this cone, and $\dot{\mathbf{x}}$ must be tangent to the cone at $\mathbf{x}$. Since the radial vector $\mathbf{x}$ is also tangent to the cone, the acceleration $\lambda \mathbf{x} \times \dot{\mathbf{x}}$ of [7.2.11] is always normal to this cone. The shapemetric k_S on $\mathbf{R}^3 \setminus \{0\}$ is conformal to the standard one so that this normality also holds in the shape metric. This demonstrates that

Every extremal trajectory is a geodesic on some cone C in $\mathbf{R}^3$. The geodesic equations are those of the shape-induced metric on the cone.

We now solve the geodesic equations on the cone. Without loss of generality, we can suppose that $\mathbf{L}_0$ is parallel to the z-axis: $\mathbf{L}_0 = e_3 \|\mathbf{L}_0\|$. Then the equation for the cone reads $r \|\mathbf{L}_0\| \cos\theta = er$, or

$$\cos\theta = \frac{e}{2} / \|\mathbf{L}_0\|. \tag{7.2.15}$$

(Note $\|\mathbf{L}_0\| \geq \frac{1}{2}|e|$, so that the right hand side of this equation is less than or equal to 1 in magnitude.) The induced metric on C is then

$$k_S = \frac{1}{4r}[dr^2 + r^2 \sin^2\theta d\varphi^2]. \tag{7.2.16a}$$

Set

$$\rho = \sqrt{r}. \tag{7.2.16b}$$

Then

$$k_S = d\rho^2 + c^2 \rho^2 d\varphi^2, \tag{7.2.16c}$$

where c^2 is the constant

$$c^2 = \frac{1}{4} \sin^2\theta = \frac{1}{4}\left(1 - \left[\frac{1}{2}\frac{e}{\|\mathbf{L}_0\|}\right]^2\right). \tag{7.2.16d}$$

This shows that the effect of the nonstandard metric k_S is to make the cone's geometric angle of opening, smaller than its opening angle as measured in the standard metric, $dr^2 + r^2 \sin^2(\theta) d\varphi^2$.

[7.2.16c] is the equation of the metric on a standard cone whose generator makes an angle $\sin^{-1}(c)$ with its axis of symmetry. In particular, the metric is flat. The geodesics can be found by cutting and unfolding the cone along a generator. The cone is isometric to the closed convex wedge

$$C^* = \{0 \leq \varphi \leq 2\pi c\} \text{ in } \mathbb{R}^2 \text{ with its standard metric,}$$

with boundary rays identified (with the generator) and (ρ, φ) being polar coordinates on this $\mathbb{R}^2$.

From here one can very easily obtain explicit coordinate expressions for the extremal trajectories of the three particle system. We will content ourselves with two observations regarding the extremals.

Observation 1. The largest possible value for c is $c = \frac{1}{2}$. This corresponds to $\theta = \pi$, which is the negative z-axis. This corresponds to C^* being a half-plane. All other values of c lead to opening cones C^* with opening angles $2\pi c < \pi$. On these cones every geodesic except the rays through the cone point (that is, the generators) are self-intersecting. These rays represent simple dilations: $(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)) = r(t)(\mathbf{x}_1(0), \mathbf{x}_2(0), \mathbf{x}_3(0))$. Except for these rays, all extremal curves have points of self-intersection, and thus define extremal loops in shape space.

Observation 2. The moment of inertia of an optimal trajectory

$$I(t) = \rho(t)^2 \text{ is a quadratic function of time.}$$

This can be seen by writing the $\rho^2 = x^2 + y^2$ where x and y are Cartesian coordinates on C^*, and then writing the parametric equation for a line in terms of x and y.

§7.3. Three Point Masses in Space.

The configuration space is

$$Q = \{m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + m_3\mathbf{x}_3 = 0\} \subset \mathbb{R}^{3\times 3}. \tag{7.3.1a}$$

The change of variables [7.1], with $\mathbf{x}_i$ in place of z_i gives

$$Q \cong \mathbb{R}^3 \times \mathbb{R}^3 \text{ with the standard metric } \|d\mathbf{q}_1\|^2 + \|d\mathbf{q}_2\|^2 \tag{7.3.1b}$$

Geometry of Shape Space. The shape space

$$
\begin{aligned}
S &= \mathbf{R}^3 \times \mathbf{R}^3/SO(3) \\
&\cong \mathbf{R}^2 \times \mathbf{R}^2/O(2), \qquad (7.3.2)
\end{aligned}
$$

where the groups act diagonally. In order to see this second isomorphism identify $\mathbf{R}^2$ with the x-y plane in $\mathbf{R}^3$. For any pair of points $(\mathbf{q}_1, \mathbf{q}_2) \in \mathbf{R}^3 \times \mathbf{R}^3$ we can find a rotation matrix $R \in SO(3)$ such that $(R\mathbf{q}_1, R\mathbf{q}_2) \in \mathbf{R}^2 \times \mathbf{R}^2 \subset \mathbf{R}^3 \times \mathbf{R}^3$, namely find an R which takes span $\{\mathbf{q}_1, \mathbf{q}_2\}$ into $\mathbf{R}^2$. The ambiguity in R is $O(2)$, not $SO(2)$.

The natural projection

$$\mathbf{R}^2 \times \mathbf{R}^2/SO(2) \to \mathbf{R}^2 \times \mathbf{R}^2/O(2) \qquad (7.3.3)$$

is 2: 1 except on the orbits of the collinear points, where it is 1: 1. Since $O(2)/S0(2) \cong Z_2$, the two-element group, we have

$$S = S^*/\mathbf{Z}_2 \qquad (7.3.4)$$

where $S^* = \mathbf{R}^2 \times \mathbf{R}^2/SO(2)$ is the planar shape space of §7.2. The $\mathbf{Z}_2$ action on $S^* = \mathbf{C} \times \mathbf{R}$ is given by $(w,t) \to (\bar{w}, t)$ where $\bar{w}$ denotes the complex conjugate of w. (This can be seen by noting that the $\mathbf{Z}_2$ action can be realized on $\mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{C} \times \mathbf{C}$ by $(z_1, z_2) \to (\bar{z}_1, \bar{z}))_2)$ and then inducing the action on S^* by the projection [7.2.5a].) Consequently, S has the structure of a closed half-space with a distinguished point on its boundary. A more accurate picture is that S is a closed convex cone.

S is stratified according to symmetry type. There are three strata, the open interior, the boundary cone, and the apex of the cone. Points in the interior of S represent generic triangles. They have non-zero area. Points on the boundary are the collinear triangles. The apex point represents the "point" triangle, in which all three masses are co-incident.

Our three point masses must remain in the plane which they define up until they become collinear, that is, until their shape hits the boundary of S. More precisely, we have

PROPOSITION 7.3.1. *Suppose that three point masses move continuosly and piecewise differentiably in space in such a way that their total angular momentum about their center of mass is zero (whenever it is defined). Suppose that $a \leq t \leq b$ is an interval of time for which they are never* *collinear. Then the plane P_t containing the three points is* <u>constant</u> *(in the center of mass frame) on any time interval $a \leq t \leq b$ for which the three points are never collinear.*

Proof. The theorem is more easily proved in terms of the coordinates $(\mathbf{q}_1, \mathbf{q}_2) \in \mathbf{R}^3 \times \mathbf{R}^3$. The plane P_t is span $\{\mathbf{q}_1(t), \mathbf{q}_2(t)\} \subset \mathbf{R}^3$. The proposition is equivalent to the implication $\mathbf{q}_1 \times \mathbf{v}_1 + \mathbf{q}_2 \times \mathbf{v}_2 = 0$ and $\mathbf{q}_1 \times \mathbf{q}_2 \neq 0$ implies $\mathbf{v}_1, \mathbf{v}_2 \in P =$ span $\{\mathbf{q}_1, \mathbf{q}\}$.

To prove this, write $\mathbf{e}_3 = \mathbf{q}_1 \times \mathbf{q}_2$. Since $\mathbf{e}_3 \neq 0, \{\mathbf{q}_1 \times \mathbf{e}_3, \mathbf{q}_2 \times \mathbf{e}_3, \mathbf{e}_3\}$ form a basis for $\mathbf{R}^3$. Write

$$\mathbf{v}_1 = \mathbf{w}_1 + a\mathbf{e}_3, \mathbf{v}_2 = \mathbf{w}_2 + b\mathbf{e}_3$$

with $\mathbf{w}_i \in P$. Then the total angular momentum $\mathbf{M}$ is

$$\begin{aligned} \mathbf{M} &= \mathbf{q}_1 \times \mathbf{v}_1 + \mathbf{q}_2 \times \mathbf{v}_2 \\ &= \lambda \mathbf{e}_3 + a\mathbf{q}_1 \times \mathbf{e}_3 + b\mathbf{q}_2 \times \mathbf{e}_3. \end{aligned}$$

Since this is zero, it follows that $a = b = 0$.

Remark. A less direct proof is obtained from Iwai's expression for the curvature and connection of the natural mechanical connection. With respect to a certain local section Ψ, both of these take values in the one-dimensional subalgebra generated by $\mathbf{e}_3 =$ span $(\Psi_1, \Psi_2)^{\perp}$. The result follows from the Ambrose-Singer Theorem.

What happens if the particles become collinear? Can the plane they define change? Yes. **See figure 1.**

In this figure $\mathbf{q}_1(t), \mathbf{q}_2(t)$ are continuous $_i$ and approach $\mathbf{q}_0$ as $t \to 0$. Both the left and right derivatives, $\dot{\mathbf{q}}_i(0^-)$ and $\dot{\mathbf{q}}_i(0^+)$, of these vector-valued functions exist at $t = 0$ and these derivatives lie in the plane perpindicular to $\mathbf{q}_0$. These right and left derivatives are not equal. The initial plane of motion is spanned by $\mathbf{q}_0$ and $\dot{\mathbf{q}}_1(0^-)(= -\dot{\mathbf{q}}_2(0^-))$. The final plane of motion is spanned by $\mathbf{q}_0$ and $\dot{\mathbf{q}}_1(0^+)$. The total angular momentum is zero throughout the motion. In summary,

<u>The collinear configurations can act as "switching yards" between different planes of motion.</u>

What are the optimal trajectories? As long as the particles are not collinear, they move in a fixed plane according to proposition 7.3.1, and so the problem is identical to the problem solved in §7.2. When they become collinear, the plane of motion can change. In summary, we have

PROPOSITION 7.3.2. *The optimal curves are piecewise smooth concatenations of the planar optimal curves of §7.2. Derivative discontinuities can only occur when the masses become collinear. If the desired re-orientation $R \in SO(3)$ does not preserve the initial plane, $P =$ span $\{\mathbf{q}_1(0), \mathbf{q}_2(0)\}$, (assuming it is a plane) then the optimal curve must have such a derivative discontinuity.*

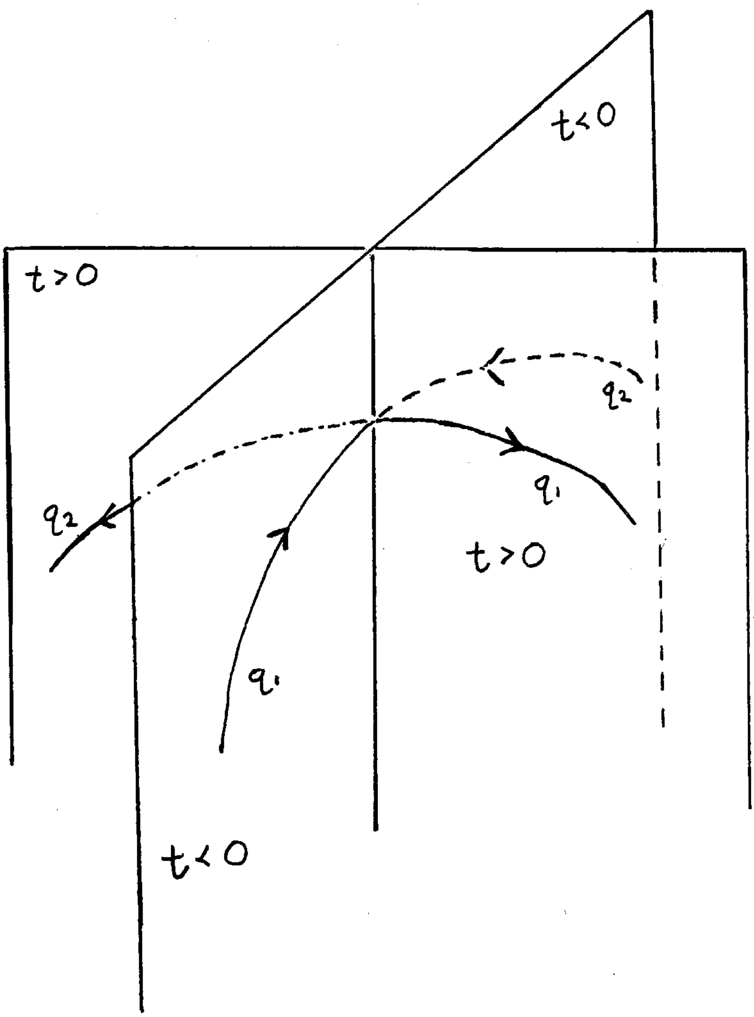

Figure 1

Appendix.

Calculation of the Momentum Map [7.2.13].

We derive the formula [7.2.13]

$$\mathbf{L} = \mathbf{x} \times \mathbf{v} + \frac{1}{2} e\mathbf{x}/r$$

for the $SO(3)$ momentum map $\mathbf{L}$ of §7.2. We will use the symbol "J" instead of "$\mathbf{L}$".

We begin by recalling the definition of a momentum map, and the standard formula for the momentum map associated to an action on configuration space. If a Lie group K acts in a Poisson fashion on the Poisson manifold P, then a *momentum map* for this action is a function

$$J\colon P \to \kappa^* = \text{ dual of Lie algebra of } K$$

which satisfies

$$\{f, J \cdot \xi\} = df \cdot \xi_P \text{ for all smooth functions } f \text{ on } P, \text{ and all } \xi \in \mathfrak{g}.$$

Here $J \cdot \xi$ is the ξ component of J, and ξ_P is the infinitesimal generator on P. If $P = T^*Q$ and the K action is the cotangent lift of an action of K on Q, then

$$(A1) \qquad J(q,p) = p \cdot \xi_Q(q)$$

defines a momentum map. (This is formula [3.1a].)

The Poisson structure on the Wong phase space $\mathfrak{g}^*(Q) \oplus T^*S$ is induced from that on T^*Q so we can use [A1] to calculate the corresponding momentum map on the Wong phase space $\mathfrak{g}^*(Q) \oplus T^*S$. Recall in that set-up (§4) we had $\pi\colon Q \to S$ a principal bundle with structure group G, and connection A. The isomorphism

$$\mathfrak{g}^* \times \pi^* T^* S \to T^* Q$$

is given by

$$(A2) \qquad (\mu, (q, p_S)) \to p = \mu \cdot A_q + h_q^* p_S.$$

Here $q \in Q$ and $p_S \in T^*_{\pi(q)}S$, and $h^*_q \colon T^*_{\pi(q)}S \to T^*_q Q$ is the dual of the horizontal lift operator h. Suppose that K acts by bundle automorphisms of Q, that is, it commutes with the G action. Then by projection K acts on S and $\pi^*\xi_Q = \xi_S$. [A1] reads

$$(A3) \qquad (J \cdot \xi)(q,p) = \mu \cdot A_q \cdot \xi_Q + p_S \cdot \xi_S.$$

(The h^*_q disappears because $\xi_Q = h \cdot \xi_S +$ vertical, and $h^*_q p_S \cdot$ vertical $= 0$.) J is automatically G-invariant, so defines a function on the Wong phase space $\mathfrak{g}^*(Q) \oplus T^*S = (\mathfrak{g}^* \times \pi^* T^* S)/G$. This is the desired momentum map. Note that the second term of [A3] is the ξ component of the standard momentum map [A1] for the K action on T^*S.

We apply formula [A3] to our $SO(3)$ action. First, lift the action to the standard action of $K = SU(2)$ on $\mathbb{C}^2$. Choose the (standard) basis

$$e_1 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \frac{i}{2}\sigma_1;$$
$$e_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \frac{i}{2}\sigma_2;$$
$$e_3 = \frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{i}{2}\sigma_3.$$

for the Lie algebra of K. (The σ_i are the Pauli matrices.) This gives us an identification $\kappa^* \cong \mathbb{R}^3 \cong so(3)^*$. Moreover, if we write $\omega = \Sigma\omega^i e_i$, then under this identification

$$\omega_S(\mathbf{x}) = \omega \times \mathbf{x}, \text{ where } \mathbf{x} \in S \cong \mathbb{R}^3.$$

It follows that the second term of [A3] is

$$p_S \cdot \omega_S(\mathbf{x}) = (\mathbf{x} \times \mathbf{p}) \cdot \omega,$$

which is the ω-component of the standard momemtum map, $\mathbf{x} \times \mathbf{p}$, for the action of $SO(3)$ on T^*S. This yields the first term of [7.2.13].

We will be done if we can show that the first term of [A3] gives the second term of [7.2.13], that is, if

$$(A4) \qquad eA_q \cdot \omega_Q = \frac{e}{2r}\mathbf{x} \cdot \omega$$

Aside: e, the fiber coordinate of $g^*(Q)$ is simply a real number, the same real number appearing in [A4]. The group $G = SO(2)$ is a one-dimensional

Abelian group. Its dual Lie algebra g^* can be identified with $\mathbf{R}$, on which $SO(2)$ acts trivially. [4.4] then reads

$$\mathfrak{g}^*(Q) \oplus T^*S = \mathbf{R} \times T^*S.$$

The $\mathbf{R}$ factor is "central", that is the function e is a Casimir: $\{e, f\} = 0$ for all functions f. But the Poisson structure depends on e. In fact, the symplectic leaves are $\{e\} \times T^*S$ with symplectic form $\omega_0 + eF$, where ω_o is the standard symplectic form on T^*S, and F is the curvature form, pulled back to T^*S by the cotangent projection.

To calculate [A4], one checks that at $(q_1, q_2) \in Q = \mathbf{C}^2$ one has

$$(A5) \qquad (e_1)_Q = \frac{1}{2}(q_2, -q_1); \quad (e_2)_Q = \frac{i}{2}(q_2, q_1); \quad (e_3)_Q = \frac{i}{2}(q_1, -q_2)$$

Recall that the connection is

$$(A6) \qquad A = \frac{1}{r} \text{ im } (\bar{q}_1 dq_1 + \bar{q}_2 dq_2)$$

Plugging [A5] into [A6] , and using formula [7.2.5] for π :

$$\pi((q_1, q_2)) = (2\bar{q}_1 q_2, |q_1|^2 - |q_2|^2) = (x + iy, z),$$

we obtain

$$(A7) \qquad A \cdot (e_1)_Q = \frac{1}{2r}x; \quad A \cdot (e_2)_Q = \frac{1}{2r}y; \quad A \cdot (e_3)_Q = \frac{1}{2r}z.$$

[A4] follows immediately from [A7].

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