Analytic Proof of Chaos in Leggett's Equations for ^{3}He

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The Leggett equations, assumed to govern the magnetization for superfluid ^{3}He, are proved to exhibit "deterministic chaos." As a corollary, this explains why Maki and Ebisawa (1976) were unable to find a complete set of integrals of the motion for the B phase in the presence of a constant magnetic field.

1. INTRODUCTION

The magnetization vector **M** for superfluid ^{3}He is described by equations of motion, the Leggett equations, similar to those for a rigid pendulum. A step change in an external magnetic field can "kick" the vector **M** into a position of unstable equilibrium analogous to the $\theta = \pi$ configuration for the pendulum.[1] The present paper analyzes the dynamics near the unstable equilibrium in order to say something about chaotic motion of the magnetization.

Chaotic behavior in a rigid pendulum

$$\begin{aligned} \dot{\theta} &= p + \varepsilon(d\theta + a \sin \omega t) \\ \dot{p} &= -(g/l) \sin \theta \end{aligned} \tag{1}$$

with damping term $\varepsilon\, d\dot{\theta}$ and forcing torque $\varepsilon a \omega \cos \omega t$ is fairly well understood (see Marsden[2]). These equations are very similar to the ^{3}He equations of Leggett

$$\left.\begin{aligned} \dot{\theta} &= \gamma M/\chi - \tfrac{1}{2}\Gamma \sin 2\theta - \gamma B(t) \\ \gamma \dot{M}/\chi &= -\tfrac{1}{2}\Omega^2 \sin 2\theta \end{aligned}\right\} \quad \text{A phase} \tag{2a}$$

$$\left.\begin{aligned} \dot{\theta} &= \gamma M/\chi + \tfrac{4}{15}\Gamma(\sin\theta)(1 + 4\cos\theta) - \gamma B(t) \\ \gamma \dot{M}/\chi &= \tfrac{4}{15}\Omega^2(\sin\theta)(1 + 4\cos\theta) \end{aligned}\right\} \quad \text{B phase} \tag{2b}$$

which were recently investigated numerically by Yamaguchi[3] for chaotic

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behavior. Here θ is an angle characterizing the order parameter, and M is the projection of the magnetization vector **M** onto the axis of the external forcing ac magnetic field

$$\mathbf{B}(t) = \mathbf{B} \sin \omega t$$

The equations are usually written in terms of the spin vector **S** and its projection S, which are related to **M** by

$$\mathbf{M} = (\gamma/\chi)\mathbf{S}$$

where γ is the gyromagnetic ratio and χ is the spin susceptibility. The quantity Γ is a temperature- and phase-dependent relaxation parameter.[4,5] Finally, Ω is the Leggett or longitudinal ringing frequency for the appropriate phase.

Equations (2a) and (2b) are the form that the Leggett equations take under certain simplifying assumptions regarding initial conditions that are supposed to be appropriate to the parallel ringing experiments of Webb *et al.*[1,5] See Brinkman and Cross (Ref. 6, pp. 148–154), Leggett and Takagi,[4] or Lee and Richardson[7] for theoretical details.

In this paper we use an analytic method to prove Yamaguchi's findings of chaotic dynamics in the ^{3}He equations. This analytic method was pioneered by Melnikov[8] and has been used on the pendulum and a variety of other problems.[2,9] Melnikov's criterion proves the existence of a type of chaotic dynamics known as a Smale horseshoe. Its existence implies, for instance, that if we mark a zero every time the pendulum bob goes to the right at the bottom of a swing and a one every time it goes to the left, then, for ε small enough, there are initial conditions for the pendulum such that the subsequent motion ticks off the binary expansion of the fractional part of any arbitrarily given number.

These chaotic initial conditions are near the infinite period orbit for the $\varepsilon = 0$ system. This orbit consists of an infinitesimal displacement of the pendulum bob from its unstable equilibrium with its subsequent infinite period return. In the phase plane (see Fig. 1) such an orbit is called a "separatrix" (it separates the running modes from the oscillatory modes) or "homoclinic orbit." The chaotic behavior can be imagined by thinking of the bob approaching the top of its orbit with a slow velocity and the periodic forcing torque turned on.

These results use perturbative techniques and hence depend on ε being small. The proper choice for ε in the ^{3}He systems seems to be

$$\varepsilon = \gamma B_0/\Omega \approx 0.01$$

where B_0 is a fixed, small magnetic field strength (Ω/γ is about 10 G). Set

$$p = \frac{\gamma}{\chi\Omega} M, \qquad \bar{\omega} = \frac{\omega}{\Omega}, \qquad A = \frac{B}{B_0} \approx 10, \qquad D = \frac{\Gamma}{\gamma B_0} = \frac{\Gamma}{\Omega}\frac{1}{\varepsilon} \approx 10 \qquad (3)$$

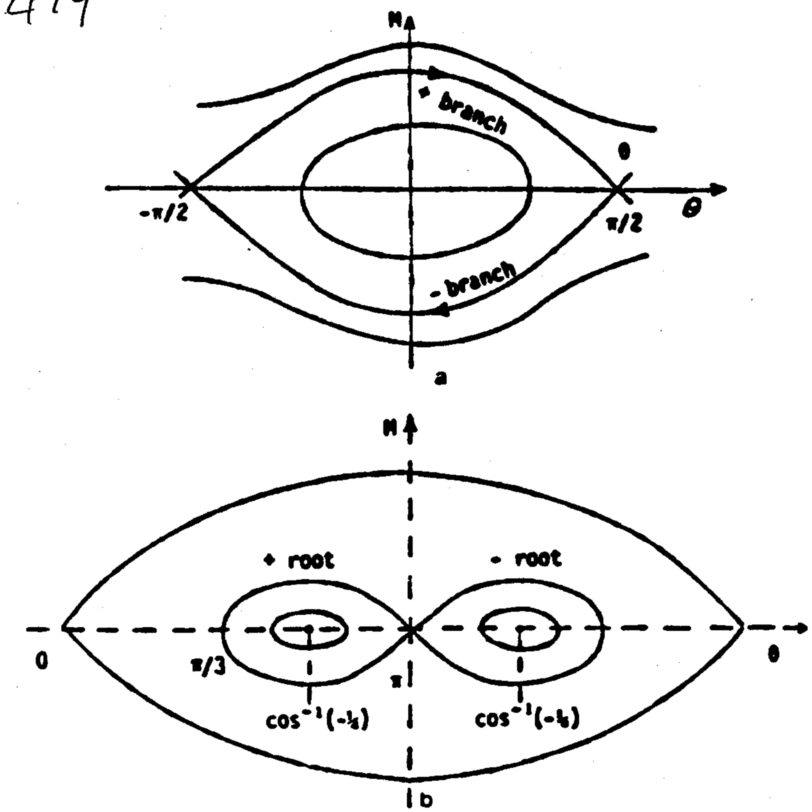

Fig. 1. Phase portraits for the unperturbed ($\varepsilon = 0$) Leggett equations. Top: A phase Bottom: B phase.

and rescale time according to

$$t \mapsto \Omega t$$

Then the ^{3}He equations in these dimensionless units are*

$$\dot{\theta} = p - \varepsilon(\tfrac{1}{2}D \sin 2\theta + A \cos \bar{\omega} t)$$
$$\dot{p} = -\tfrac{1}{2} \sin 2\theta \qquad (2$$

$$\dot{\theta} = p + \varepsilon[\tfrac{4}{15}D(\sin\theta)(1 + 4\cos\theta) - A \cos \bar{\omega} t]$$
$$\dot{p} = \tfrac{4}{15}(\sin\theta)(1 + \cos\theta) \qquad (2$$

*We compared (2a') to the data in Webb *et al.*[5] In this paper they compared their obser frequencies to the frequencies predicted by the nondissipative theory, i.e., (2a') with $A = D$. A large discrepancy was found near the theoretical zero frequency, i.e., the separatrix. numerically integrated (2a') with no forcing ($A = 0$) and various values of damping $\varepsilon D = \Gamma$ For $0 < \varepsilon D < 1.0$ the frequencies predicted by (2a) were in closer agreement with the exp mental data than the nondissipative theory, with the agreement fairly close for $\varepsilon D = 0.5$

In the next section we outline the proof of chaos for Eqs. (2a′) and (2b′). We have also proved the existence of chaos in the full three-dimensional B-phase equations as investigated by Maki and Ebisawa.[10] This is the case of constant nonzero magnetic field with the order parameter's rotation axis n *not* parallel to the magnetic field [to get Eqs. (2b), n parallel is assumed]. As a consequence of the chaotic behavior it can easily be seen that there is no other analytic integral of the motion besides the energy and Maki and Ebisawa's p_1. The proof consists in adapting the Melnikov method to this situation and is along the lines of Holmes and Marsden.[11] However, the details are complicated and so will be included in another publication.

2. MELNIKOV'S CRITERION AND THE PROOF OF CHAOS

For both the helium and the pendulum equations the unperturbed equations are in Hamiltonian form:

$$\dot{\theta} = \partial H_0/\partial p, \qquad \dot{p} = -\partial H_0/\partial \theta \tag{5}$$

with

$$H_0(\theta, p) = \tfrac{1}{2}p^2 + U(\theta)$$

and

$$U(\theta) = \begin{cases} -\cos\theta & \text{for the pendulum} \\ -\frac{1}{2}\cos^2\theta & \text{for } ^3\text{He-A} \\ \frac{1}{30}(1+4\cos\theta)^2 & \text{for } ^3\text{He-B} \end{cases}$$

The potential $U(\theta)$ is graphed in Fig. 2.

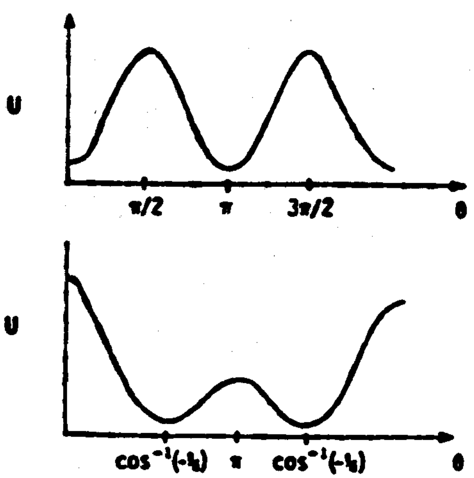

Fig. 2. Potentials for the Leggett equations. A phase, top; B phase, bottom.

To use the Melnikov method, one must have separatrices for (5). These are found by first finding saddle points, i.e., unstable equilibria for H_0. These can be read off the graphs of the U. The saddle points are given by

$$\theta = \theta_{max} = \begin{cases} \pi & \text{for the pendulum} \\ -\pi/2, \pi/2 & \text{for } ^3\text{He-A} \\ 0, \pi & \text{for } ^3\text{He-B} \end{cases}$$

and $p = 0$. Then one solves the differential equation

$$H(\theta, p) = H(\theta_{max}, 0)$$

Upon using the first of Eqs. (5), $\dot{\theta} = p$, this becomes

$$\dot{\theta} = \pm\{2[U(\theta_{max}) - U(\theta)]\}^{1/2}$$

The solutions are

$$\theta(t) = \begin{cases} 2\tan^{-1}(\sinh t) & \text{for the pendulum} \\ 2\tan^{-1}(e^{\pm t}) \pm \pi/2 & \text{for } ^3\text{He-A} \\ 2\tan^{-1}[\pm\sqrt{\tfrac{5}{3}}\cosh(t/\sqrt{5})] & \text{for } ^3\text{He-B} \end{cases} \tag{6}$$

Recall that the time has been made dimensionless to Ωt here [see Eq. (3)]. In the A phase the plus and minus forms can be specified independently. In the B phase we have only solved for the separatrices containing the stable equilibrium $\theta = \pm\cos^{-1}(\frac{1}{4})$, $p = 0$. These are the separatrices through $\theta = \pi$, $p = 0$. See Fig. 1 for the phase diagrams.

The Melnikov method reduces the proof of chaos to the computation of the integral

$$M(t_0) = \int_{-\infty}^{\infty} \det \begin{vmatrix} \partial H/\partial p & X_\theta \\ -\partial H/\partial \theta & X_p \end{vmatrix} (\theta(t - t_0), p(t - t_0))\, dt$$

along the separatrix. Here X_θ and X_p are the perturbing terms on the right-hand side of the $\dot{\theta}$ and $\dot{p}$ equations, respectively. In our examples $X_p = 0$ and $X_\theta = -D\,\partial U/\partial\theta - A\cos\bar{\omega}t$. Thus

$$M(t_0) = D\int_{-\infty}^{\infty} \left(\frac{\partial U}{\partial \theta}\right)^2 dt + A\int_{-\infty}^{\infty} \frac{\partial U}{\partial \theta}\sin\bar{\omega}t\, dt \tag{7}$$

where $\partial U/\partial\theta$ is evaluated at $\theta(t - t_0)$. This function M of the retardation time t_0 is called the Melnikov function. We now state:

The Melnikov criterion: If M has a simple zero, then chaos (in the sense of Smale horseshoes) exists in the perturbed system, for ε small. (For more details, see Guckenheimer and Holmes[9] or Marsden.[2])

To calculate the Melnikov function (6), set

$$F(t) = \frac{\partial U}{\partial \theta}\bigg|_{\theta(t)} \tag{8}$$

Then

$$M(t_0) = D \int_{-\infty}^{\infty} F(t)^2\, dt + A\, \mathrm{Im}\left[\int_{-\infty}^{\infty} F(t - t_0)\, e^{i\omega t}\, dt\right]$$

The first term is a constant. The second term can be rewritten

$$A\, \mathrm{Im}\left[e^{i\omega t_0} \int_{-\infty}^{\infty} F(t)\, e^{i\omega t}\, dt\right] \tag{9}$$

demonstrating that it is oscillatory. The Melnikov criterion that this have simple zeros is then that the constant term is less than the oscillatory amplitude:

$$D \int_{-\infty}^{\infty} F(t)^2\, dt < A|\hat{F}(\omega)| \tag{10}$$

where $\hat{F}(\omega)$ is the integral in (9). In more physical terms this criterion says that the forcing must be sufficiently greater than the damping to ensure chaos. Lengthy calculations involving residues yield

$$\hat{F}(\bar{\omega}) = \begin{cases} \pm(i\pi/8)\bar{\omega}\, \mathrm{sech}\,(\tfrac{1}{2}\pi\bar{\omega}) & \text{for the A phase} \\ \pm 8\pi\bar{\omega}\, \mathrm{sech}\,(\sqrt{5}\,\pi\bar{\omega})\, \sin[\sqrt{5}\log(2-\sqrt{3})\bar{\omega}] & \text{for the B phase} \end{cases}$$

where the plus and minus refer to the separatrix over which one is traveling. Also,

$$\int_{-\infty}^{\infty} F(t)^2\, dt = \begin{cases} \dfrac{2}{3} & \text{for the A phase} \\ \left(\dfrac{4}{15}\right)^2 \sqrt{5} \cdot 4108 \cdot \dfrac{1}{72} \log\left(1 + \dfrac{25}{8\sqrt{3}} \log \dfrac{7+4\sqrt{3}}{7-4\sqrt{3}}\right) & \\ \approx 2.505 & \text{for the B phase} \end{cases}$$

The Melnikov criterion then guarantees the existence of chaos provided

$$\frac{\Gamma}{\gamma}\frac{2}{3} < B \frac{\pi}{8}\frac{\omega}{\Omega}\, \mathrm{sech}\left(\frac{\pi}{2}\frac{\omega}{\Omega}\right) \qquad \text{in the A phase}$$

$$\frac{\Gamma}{\gamma} \cdot 2.505 < B 8\pi \frac{\omega}{\Omega} \sin\left[\sqrt{5}\log(2-\sqrt{3})\frac{\omega}{\Omega}\right] \mathrm{sech}\left(\sqrt{5}\,\pi\frac{\omega}{\Omega}\right)$$

in the B phase

These criteria agree very roughly with the regions of the $(\bar{\omega}, \gamma B/\Omega)$ parameter plane in which Yamaguchi (Ref. 2, Fig. 2) numerically found chaotic behavior. Any better than a rough agreement is not to be expected, for the quantitative relationships between the Melnikov criterion and the various numerical measures of chaos are complicated and poorly understood.

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