Examples of Singular Reduction

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Introduction

The construction of the reduced space for a symplectic manifold with symmetry, as formalized by Marsden and Weinstein [13], has proved to be very useful in many areas of mathematics ranging from classical mechanics to algebraic geometry. In the ideal situation, which requires the value of the moment map to be weakly regular, the reduced space is again a symplectic manifold. A lot of work has been done in the last ten years in the hope of finding a 'correct' reduction procedure in the case of singular values. For example, Arms, Gotay and Jennings describe several approaches to reduction in [4]. At some point it has also been observed by workers in the field that in all examples the level set of a moment map modulo the appropriate group action is a union of symplectic manifolds. Recently Otto has proved that something similar does indeed hold, namely that such a quotient is a union of symplectic *orbifolds* [16]. Independently two of us, R. Sjamaar and E. Lerman, have proved a stronger result [21]. We proved that in the case of proper actions the reduced space, which we simply took to be the level set modulo the action, is a stratified symplectic space. Thereby we obtained a global description of the possible dynamics, a procedure for lifting the dynamics to the original space and a local characterization of the singularities of the reduced space. (The precise definitions will be given below.) The goal of this paper is twofold. First of all, we would like to present a number of examples that illustrate the general theory. Secondly, in computing the examples we have noticed that many familiar methods for computing reduced spaces work nicely in the singular situations. For instance, in the case of a lifted action on a cotangent bundle the reduced space at the zero level is the 'cotangent bundle' of the

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orbit space. And in some cases the reduced space can be identified with the closure of a coadjoint orbit.

1 A Simple Example

Consider the standard action of the circle group $SO(2)$ on $\mathbf{R}^2$, and lift this action to $T^*\mathbf{R}^2 \simeq \mathbf{R}^2 \times \mathbf{R}^2$. In coordinates,

$$\begin{pmatrix} q^1 \\ q^2 \\ p_1 \\ p_2 \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} \cos\theta & -\sin\theta & \multicolumn{2}{c}{0} \\ \sin\theta & \cos\theta & & \\ \hline \multicolumn{2}{c|}{0} & \cos\theta & -\sin\theta \\ & & \sin\theta & \cos\theta \end{array}\right) \begin{pmatrix} q^1 \\ q^2 \\ p_1 \\ p_2 \end{pmatrix},$$

and the canonical symplectic form is $\omega = dq^1 \wedge dp_1 + dq^2 \wedge dp_2$. The corresponding momentum map J is the angular momentum $J(q,p) = q^1 p_2 - q^2 p_1$. Zero is a singular value of J. Let us compute the reduced space at zero, $(T^*\mathbf{R}^2)_0$, which we will take to be the quotient $J^{-1}(0)/SO(2)$. The zero level set $J^{-1}(0)$ is a union of a point, 0, and of a hypersurface

$$Z = \{\, q^1 p_2 - q^2 p_1 = 0 : (q^1, q^2, p_1, p_2) \neq 0 \,\}.$$

The hypersurface is a $SO(2)$-invariant coisotropic submanifold of $T^*\mathbf{R}^2$. The group $SO(2)$ acts freely on Z and the null directions of the restriction of the symplectic form ω to Z are precisely the orbital directions (just as in the regular case). Consequently the quotient $C_1 = Z/SO(2)$ is a symplectic manifold. The other piece of the zero level set, the origin 0, is fixed by the action of $SO(2)$ and we may consider the quotient $C_0 = \{0\}/SO(2)$ as a zero-dimensional symplectic manifold. Thus the reduced space $(T^*\mathbf{R}^2)_0$ is a disjoint union of two symplectic manifolds,

$$(T^*\mathbf{R}^2)_0 = C_0 \coprod C_1. \tag{1}$$

Let us give a more concrete description of the reduced space. We claim that C_1 is $\mathbf{R}^2 \setminus \{0\}$ with the standard symplectic structure and that the reduced space as a whole is diffeomorphic to the orbifold $\mathbf{R}^2/\mathbf{Z}_2$, where the action of $\mathbf{Z}_2$ is generated by the reflection $(x^1, x^2) \to (-x^1, -x^2)$.

1.1 Digression: Smooth Structures on Reduced Spaces

Let us explain what is meant by $(T^*\mathbf{R}^2)_0$ being diffeomorphic to $\mathbf{R}^2/\mathbf{Z}_2$. In general, let (M, ω) be a Hamiltonian G-space with corresponding moment map $\Phi : M \to \mathfrak{g}^*$ and let us assume that G acts properly on M. (In all the examples that follow the group G is going to be compact and for compact groups the properness of the action is automatic.) For us the reduced space at zero, M_0, is the topological space formed by dividing the zero level set $\Phi^{-1}(0)$ by the group action, i.e.,

$$M_0 = \Phi^{-1}(0)/G.$$

(We will see later that M_0 has a lot of structure, not just a topology.) As we have just seen, $\Phi^{-1}(0)$ need not be a manifold and the action of G on the zero level set need not be free. Thus there is no reason for the reduced space so defined to be a manifold (or even an orbifold). However, as Arms et al. have observed [3], it makes sense to single out a certain subset of the set of continous functions on M_0 as follows. Call a function $f : M_0 \to \mathbf{R}$ *smooth* if there exists a smooth G-invariant function $\tilde{f}$ on M whose restriction to the zero level set $\Phi^{-1}(0)$ equals the pullback of f to $\Phi^{-1}(0)$ by the orbit map $\pi : \Phi^{-1}(0) \to \Phi^{-1}(0)/G = M_0$, i.e.,

$$\tilde{f}|_{\Phi^{-1}(0)} = \pi^* f.$$

Let us denote the set of smooth functions by $C^\infty(M_0)$. A map $F : M_0 \to N$, where N is a manifold (or an orbifold, or another reduced space), is *smooth* if for any function $\phi \in C^\infty(N)$ the pullback $F^*\phi$ is a smooth function on M_0, $\phi \circ F \in C^\infty(M_0)$. It is now clear what we mean by two singular spaces being diffeomorphic.

1.1. REMARK. If G is a discrete group acting symplectically on a manifold (M, ω), it makes sense to define the corresponding moment map to be the zero map, since the Lie algebra of G is trivial. The reduced space is then a symplectic orbifold M/G. (See [18] or [15] for the definition of an orbifold.) For example, the action of $\mathbf{Z}_2$ on $\mathbf{R}^2$ described above preserves the standard symplectic form $dx^1 \wedge dx^2$ and the reduced space is the symplectic orbifold $\mathbf{R}^2/\mathbf{Z}_2$ with ring of smooth functions isomorphic to the collection of the smooth *even* functions on $\mathbf{R}^2$.

1.2 The Reduced Space $(T^*\mathbf{R}^2)_0$ as an Orbifold

Let us now go back to our example. Consider the 2-plane

$$\Lambda = \{\, (q^1, q^2, p_1, p_2) \in T^*\mathbf{R}^2 : q^2 = 0,\ p_2 = 0 \,\}.$$

This plane is symplectic, it is completely contained in the zero level set of the moment map J and the $SO(2)$-orbit of any point $(q,p) \in J^{-1}(0)$ intersects Λ in exactly two points. Indeed, a point (q,p) lies in the zero level set if and only if q and p are collinear as vectors in $\mathbf{R}^2$. Consequently, $J^{-1}(0)/SO(2)$ is homeomorphic to $\Lambda/\mathbf{Z}_2$.

What about the two smooth structures? Clearly any $SO(2)$-invariant function on $T^*\mathbf{R}^2$ restricts to a $\mathbf{Z}_2$-invariant function on Λ. So the map

$\Lambda/\mathbf{Z}_2 \to J^{-1}(0)/SO(2)$ is smooth. To show that this map is a diffeomorphism it suffices to prove that any (smooth) $\mathbf{Z}_2$-invariant function on Λ extends to a (smooth) $SO(2)$-invariant function on $T^*\mathbf{R}^2$. By Schwarz's theorem [20] any smooth $\mathbf{Z}_2$-invariant function on Λ is a smooth function of the invariants $(q^1)^2$, ${p_1}^2$ and $q^1 p_1$ (these functions are a set of generators of the $\mathbf{Z}_2$-invariant polynomials on Λ). Now $(q^1)^2$ is the restriction to Λ of the $SO(2)$-invariant $(q^1)^2 + (q^2)^2$. Similarly,

$$\begin{aligned} {p_1}^2 &= ({p_1}^2 + {p_2}^2)\big|_\Lambda \\ q^1 p_1 &= (q^1 p_1 + q^2 p_2)\big|_\Lambda . \end{aligned}$$

Consequently the map $J^{-1}(0)/SO(2) \to \Lambda/\mathbf{Z}_2$ is smooth as well and, therefore, the two reduced spaces are diffeomorphic.

Note that the $\mathbf{Z}_2$-invariant functions on Λ form a Poisson subalgebra of $C^\infty(\Lambda)$. So the smooth functions on the reduced space $(T^*\mathbf{R}^2)_0$ form a Poisson algebra. This is an example of the fact proved by Arms et al. (loc. cit.) that the set of smooth functions on a reduced space M_0 has a well-defined Poisson bracket induced by the bracket on the original manifold M.

The Poisson bracket of $C^\infty((T^*\mathbf{R}^2)_0)$ is compatible with the symplectic structure of the pieces C_1 and C_0 of the reduced space (see (1)) in the following sense. A pair of functions f and g in $C^\infty((T^*\mathbf{R}^2)_0)$ restrict to a pair of smooth functions on the symplectic manifold C_1. The symplectic structure of C_1 defines a Poisson bracket $\{\cdot,\cdot\}_{C_1}$. It is easy to check that this new bracket coincides with the bracket induced by the Poisson structure on $C^\infty((T^*\mathbf{R}^2)_0)$, i.e.,

$$\{f|_{C_1}, g|_{C_1}\}_{C_1} = \{f,g\}_{(T^*\mathbf{R}^2)_0}\big|_{C_1} .$$

Similarly, one can show that

$$\{f,g\}_{(T^*\mathbf{R}^2)_0}\big|_{C_0} = 0,$$

which is consistent with viewing C_0 as a zero-dimensional symplectic manifold. We thus see that the Poisson bracket of $C^\infty((T^*\mathbf{R}^2)_0)$ and the decomposition (1) of the reduced space into symplectic manifolds are intimately related.

1.3 Reduction via Invariants

Let us present a different calculation of the reduced space $(T^*\mathbf{R}^2)_0$. The calculation uses invariant theory, an approach advocated by R. Cushman. We will realize the reduced space as a subspace of $\mathbf{R}^4$ cut out by the equations

$$\left\{ \begin{aligned} {x_1}^2 &= {x_2}^2 + {x_3}^2 \\ x_4 &= 0 \\ x_1 &\geq 0 \end{aligned} \right. \tag{2}$$

In words, this reduced space is diffeomorphic to the top half, with vertex included, of the standard cone in $\mathbf{R}^3$. Consider a change of variables

$$\left\{ \begin{aligned} u_1 &= \tfrac{1}{2}(q^2 - p_1) \\ u_2 &= \tfrac{1}{2}(q^1 - p_2) \\ u_3 &= \tfrac{1}{2}(q^1 + p_2) \\ u_4 &= \tfrac{1}{2}(q^2 + p_1) \end{aligned} \right.$$

and set

$$\left\{ \begin{aligned} z_1 &= u_1 + iv_1 \\ z_2 &= u_2 + iv_2 . \end{aligned} \right.$$

We have thus identified $T^*\mathbf{R}^2$ with $\mathbf{C}^2$. In these complex coordinates the symplectic form is given by $\omega = i\,(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$, the action of the circle group $SO(2) \simeq U(1)$ by

$$e^{i\theta} \cdot (z_1, z_2) = (e^{-i\theta} z_1, e^{i\theta} z_2)$$

and the moment map by $J(z_1,z_2) = |z_2|^2 - |z_1|^2$. It is easy to see that the set of (real) invariant polynomials is generated by four polynomials:

$$\begin{aligned} \sigma_1 &= |z_2|^2 + |z_1|^2, \\ \sigma_2 &= z_1 z_2 + \overline{z_1 z_2}, \\ \sigma_3 &= i(z_1 z_2 - \overline{z_1 z_2}), \\ \sigma_4 &= |z_2|^2 - |z_1|^2 . \end{aligned}$$

The map $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) : \mathbf{C}^2 \to \mathbf{R}^4$ pushes down to an injective map $\bar{\sigma} : \mathbf{C}^2/SO(2) \to \mathbf{R}^4$. The invariants satisfy the relations

$$\left\{ \begin{aligned} \sigma_1^2 - \sigma_4^2 &= \sigma_2^2 + \sigma_3^2 \\ \sigma_1 &\geq 0. \end{aligned} \right. \tag{3}$$

Consequently the image of $\bar{\sigma}$ is a subset of $\mathbf{R}^4$ cut out by the equations

$$\left\{ \begin{aligned} {x_1}^2 &= {x_2}^2 + {x_3}^2 \\ x_1 &\geq 0 \end{aligned} \right.$$

Therefore the reduced space $(T^*\mathbf{R}^2)_0 := \{\sigma_4 = 0\}/SO(2)$ embeds in $\mathbf{R}^4$ as the subset cut out by (2) as claimed. If we ignore the fourth coordinate, we see that the reduced space is simply a round cone in $\mathbf{R}^3$. Since the invariants

$\sigma_1, \ldots, \sigma_4$ are quadratic, their linear span in $C^\infty(\mathbf{C}^2)$ forms a four-dimensional Lie algebra under the standard Poisson bracket. Alternatively, it is enough to note that

$$\begin{aligned} \{\sigma_i, \sigma_4\} &= 0 \qquad \text{for } i = 1, \ldots, 4 \\ \{\sigma_1, \sigma_2\} &= 2\sigma_3 \\ \{\sigma_1, \sigma_3\} &= -2\sigma_2 \\ \{\sigma_2, \sigma_3\} &= 2\sigma_1. \end{aligned}$$

Therefore, the correspondence

$$\sigma_4 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 \leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_3 \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

establishes an isomorphism between the Lie algebra spanned by the generators of the invariants and $\mathfrak{gl}(2, \mathbf{R})$. The image cut out by (2) is nothing more than half of the nilpotent cone, the closure of the connected component of the principal nilpotent orbit in $\mathfrak{gl}(2, \mathbf{R})$.

More intrinsically this can be seen as follows. The moment map for the action of $Sp(T^*\mathbf{R}^2, \omega) \simeq Sp(2, \mathbf{R})$ on $T^*\mathbf{R}^2 \simeq \mathbf{R}^4$ identifies $\mathfrak{sp}(2, \mathbf{R})$ with the Poisson algebra of quadratic polynomials. The polynomials that commute with σ_4 then get identified with $\mathfrak{u}(1,1)$, which is isomorphic to $\mathfrak{gl}(2, \mathbf{R})$. We will come back to this point in Section 5, Remark 5.4.

2 A Summary of the General Theory

The goal of this section is to introduce the notion of a stratified symplectic space, to explain how this notion arises naturally in reduction and to describe some properties of reduced spaces.

2.1 Stratifications

The main idea of a stratification is that of a partition of a nice topological space into a disjoint union of manifolds. Thus a manifold is trivially a stratified space. A more interesting example of a stratified space is that of a *cone on a manifold*: given a manifold M the open cone $\mathring{C}M$ on M is the product $M \times [0, \infty)$ modulo the relation $(x, 0) \sim (y, 0)$ for all $x, y \in M$. That is, $\mathring{C}M$ is $M \times [0, \infty)$ with the boundary collapsed to a point, the vertex $*$ of the cone. The cone $\mathring{C}M$ is a disjoint union of two manifolds: $M \times (0, \infty)$ and the vertex $*$. Similarly one can consider the cone $\mathring{C}(\mathring{C}M)$ on the cone $\mathring{C}M$,

$$\mathring{C}(\mathring{C}M) = \left(\mathring{C}M \times [0, \infty)\right) / \sim .$$

The space $\mathring{C}(\mathring{C}M)$ is a union of three manifolds:

the vertex $*$ of $\mathring{C}(\mathring{C}M)$;
the open half line $\{*\} \times (0, \infty)$ through the vertex of $\mathring{C}M$;
the manifold $(M \times (0, \infty)) \times (0, \infty)$.

In general we will see that locally a stratified space is a cone on a cone on a cone .... Let us now make this precise.

2.1. Definition. A *decomposed space* is a Hausdorff paracompact topological space X equipped with a locally finite partition $X = \coprod_{i \in \mathcal{I}} S_i$ into locally closed subsets S_i called *pieces*, each of which is a manifold.

We shall only consider decompositions each of whose pieces has the structure of a *smooth* manifold. A given space may be decomposed in a number of different ways.

2.2. Example. Consider the subset of $\mathbf{R}^2$

$$Y = \{ (x^1, x^2) \in \mathbf{R}^2 : x^2 = 0 \} \cup \{ (x^1, x^2) \in \mathbf{R}^2 : x^1 \geq 0, x^2 \geq 0 \}.$$

The space Y can be broken up into a union of manifolds as

$$Y = \{x^2 = 0\} \cup \{x^1 > 0, x^2 > 0\} \cup \{x^1 = 0, x^2 > 0\} \tag{4}$$

or as

$$\begin{aligned} Y = & \{x^1 > 0, x^2 > 0\} \cup \{x^1 = 0, x^2 > 0\} \cup \{(0,0)\} \\ & \cup \{x^1 < 0, x^2 = 0\} \cup \{x^1 > 0, x^2 = 0\}. \end{aligned} \tag{5}$$

2.3. Example. A triangulated space is a decomposed space, if we declare the strata to be the (combinatorial) interiors of the simplexes.

2.4. Example. If $X = \coprod_{i \in \mathcal{I}} S_i$ is a decomposed space, the cone $\mathring{C}X$ has a natural decomposition

$$\mathring{C}X = \{*\} \cup \coprod_{i \in \mathcal{I}} S_i \times (0, \infty).$$

2.5. Example. The product of two decomposed spaces $X = \coprod S_i$ and $Y = \coprod P_j$ is a decomposed space

$$X \times Y = \coprod_{i,j} S_i \times P_j.$$

Define the *dimension* of a decomposed space X to be $\dim X = \sup_{i \in \mathcal{I}} \dim S_i$. We shall only consider finite-dimensional spaces. A stratification is a particular kind of decomposition. Its definition is recursive on the dimension of a decomposed space.

2.6. Definition (cf. [7]). A decomposed space $X = \{S_i\}_{i \in \mathcal{I}}$ is called a *stratified space* if the pieces of X, called *strata*, satisfy the following local condition:

Given a point x in a piece S there exist an open neighbourhood U of x in X, an open ball B around x in S, a compact stratified space L, called the *link* of x, and a homeomorphism $\varphi : B \times \mathring{C}L \to U$ that preserves the decomposition, i.e., maps pieces onto pieces.

2.7. Remark. We say that a decomposed space X satisfies the *condition of the frontier* if the closure of each piece is a union of connected components of pieces of X. It follows easily from Definition 2.6 that stratified spaces satisfy the condition of the frontier.

2.8. Example. The decomposition (5) satisfies the frontier condition while (4) does not. So decomposition (4) is not a stratification. We leave it to the reader to check that decomposition (5) is a stratification.

2.9. Example. A triangulated space is stratified by the interiors of its simplexes. The proof is an elementary exercise in PL-topology.

We are now in a position to define a stratified symplectic space.

2.10. Definition. A *stratified symplectic space* is a stratified space X together with a distinguished subalgebra $C^\infty(X)$ (a *smooth structure*) of the algebra of continuous functions on X such that:

(i) each stratum S is a symplectic manifold;
(ii) $C^\infty(X)$ is a Poisson algebra;
(iii) the embeddings $S \hookrightarrow X$ are Poisson.

Condition (iii) means that given two functions $f, g \in C^\infty(X)$ their restrictions, $f|_S$ and $g|_S$, to a stratum S are smooth functions on S and their Poisson bracket at the points of S coincides with the Poisson brackets of the restrictions defined by the symplectic structure on S: $\{f,g\}\big|_S = \{f|_S, g|_S\}_S$.

2.11. Theorem (cf. [21]). *Let (M, ω) be a Hamiltonian G-space with moment map $J : M \to \mathfrak{g}^*$ and suppose that the action of the Lie group G is proper. Then given an orbit $\mathcal{O} \in \mathfrak{g}^*$ the reduced space $M_\mathcal{O} := J^{-1}(\mathcal{O})/G$ is a stratified symplectic space.*

2.12. Theorem (loc. cit.). *Assume that the level set $J^{-1}(\mathcal{O})$ is connected. Then the reduced space $M_\mathcal{O}$ has a unique open stratum. It is connected and dense.*

2.13. Remark. We note two important cases when the level set is connected. First, if M is a symplectic vector space and G acts linearly on M then the zero level set is conical and so is connected. Secondly, F. Kirwan has proved [11] that if the moment map J is proper (for example if M is compact) and M is connected then the zero level set $J^{-1}(0)$ is connected. It follows then from the shifting trick, Proposition 2.16 below, that the level set $J^{-1}(\mathcal{O})$ is connected for any compact orbit $\mathcal{O}$.

The symplectic structure on the dense open stratum determines the Poisson structure on the whole reduced space and, therefore, the symplectic structures on all the lower-dimensional strata by condition (iii) of Definition 2.10. We will refer to the dense open stratum as the *top* stratum. Condition (i) also has some interesting consequences. Suppose that the top stratum is two-dimensional as in Section 1. Then all the other strata are zero-dimensional, i.e., they are isolated points. There is a temptation in view of Theorem 2.12 to discard all the lower-dimensional strata. We will see in the next section that giving in to such a temptation leads to a loss of interesting information.

2.2 Hamiltonian Mechanics on a Stratified Symplectic Space

Just as we defined in Section 1.1 a diffeomorphism between two reduced spaces, one can define an isomorphism between two stratified symplectic spaces.

2.14. Definition. Let X and Y be two stratified symplectic spaces. A map $\phi : X \to Y$ is an *isomorphism* if ϕ is a homeomorphism and the pullback map $\phi^* : C^\infty(Y) \to C^\infty(X)$, $f \mapsto f \circ \phi$ is an isomorphism of Poisson algebras.

Note that we do not explicitly require that ϕ be strata-preserving. The reason for this is that the stratification of a stratified symplectic space X is completely determined by the Poisson algebra structure on the space of smooth functions on X, as we shall see shortly.

2.15. Example (the 'shifting trick'). Let M be a Hamiltonian G-space with momentum map $J : M \to \mathfrak{g}^*$ and let $\mathcal{O}$ be any coadjoint orbit of G. Consider the symplectic manifold $M \times \mathcal{O}^-$, the symplectic product of M with the coadjoint orbit $\mathcal{O}$, endowed with the opposite of the Kirillov symplectic form. The diagonal action of G on $M \times \mathcal{O}^-$ is Hamiltonian with momentum map $J_\mathcal{O}$ given by $J_\mathcal{O}(m, \nu) = J(m) - \nu$. It is easy to check that the cartesian projection

$\Pi : M \times \mathcal{O}^- \to M$ restricts to an equivariant bijection $J_{\mathcal{O}}^{-1}(0) \cong J^{-1}(\mathcal{O})$. As a result, Π descends to a bijection between reduced spaces,

$$\tilde{\Pi} : (M \times \mathcal{O}^-)_0 \xrightarrow{\sim} M_{\mathcal{O}}.$$

2.16. PROPOSITION. *Assume that the orbit $\mathcal{O}$ is a closed subset of $\mathfrak{g}^*$. Then the map $\tilde{\Pi}$ is an isomorphism of stratified symplectic spaces.*

See [5] for a proof.

2.17. DEFINITION. A *flow* $\{\phi_t\}$ on a stratified symplectic space X is a one-parameter family of isomorphisms $\phi_t : X \to X$, $t \in \mathbf{R}$, such that $\phi_{t+s} = \phi_t \circ \phi_s$ for all t and s.

2.18. DEFINITION. Let h be a smooth function on a stratified symplectic space X, $h \in C^\infty(X)$. A *Hamiltonian flow* of h is a flow $\{\phi_t\}$ having the property that for any function $f \in C^\infty(X)$

$$\frac{d}{dt}(f \circ \phi_t) = \{f, h\} \circ \phi_t. \tag{6}$$

This is Heisenberg's form of Hamilton's equations. Since the space X is not necessarily a manifold, (6) cannot be reduced to a system of ordinary differential equations. For this reason the existence and uniqueness of the Hamiltonian flow is not immediately obvious. If X is a reduced space, the Hamiltonian flow does indeed exist and is unique [21]. Moreover, the following lemma holds.

2.19. LEMMA (cf. [21]). *Let $M_{\mathcal{O}}$ be the reduced space of a Hamiltonian G-space M at a coadjoint orbit $\mathcal{O}$ of G. The Hamiltonian flow of a smooth function $h \in C^\infty(M_{\mathcal{O}})$ preserves the stratification. The restriction of the flow of h to a stratum S equals the Hamiltonian flow of the restriction $h|_S$.*

The connected components of the strata are the symplectic leaves of $M_{\mathcal{O}}$, i.e., given any pair of points p, q in a connected component of a stratum of $M_{\mathcal{O}}$, there exists a piecewise smooth path joining p to q, consisting of a finite number of Hamiltonian trajectories of smooth functions on $M_{\mathcal{O}}$. Thus the Poisson structure of $C^\infty(M_{\mathcal{O}})$ determines the stratification of $M_{\mathcal{O}}$.

2.20. REMARK. It follows that a zero-dimensional stratum of the reduced space $M_{\mathcal{O}}$ is automatically a fixed point of *any* Hamiltonian flow. Thus the zero-dimensional strata of $M_{\mathcal{O}}$ determine relative equilibria in the original space M.

2.3 Orbit Types

We now explain where the stratification of a reduced space comes from and how it can be computed. Let G be a Lie group acting properly on a manifold M. (For example if G is compact then its action is automatically proper.) For a subgroup H of G denote by $M_{(H)}$ the set of all points whose stabilizer is conjugate to H,

$$M_{(H)} = \{\, m \in M : G_m \text{ is conjugate to } H \,\}.$$

By virtue of the slice theorem for proper actions (see e.g. Palais [17]), the set $M_{(H)}$ is a smooth submanifold of M, called the manifold of *orbit type* (H). Thus we have a decomposition $M = \coprod_{H<G} M_{(H)}$ of M into a disjoint union of manifolds. Theorem 2.11 can now be restated as follows.

2.21. THEOREM. *Let (M, ω) be a Hamiltonian G-space with moment map $J : M \to \mathfrak{g}^*$ and let $\mathcal{O}$ be a coadjoint orbit of G. Assume that the action of G on $J^{-1}(\mathcal{O})$ is proper. Then the intersection of the preimage of the orbit $J^{-1}(\mathcal{O})$ with a manifold of the form $M_{(H)}$, $H < G$, is a manifold. The orbit space*

$$(M_{\mathcal{O}})_{(H)} := (J^{-1}(\mathcal{O}) \cap M_{(H)})/G$$

is also a manifold. There exists a unique symplectic form $\omega_{(H)}$ on $(M_{\mathcal{O}})_{(H)}$ such that the pullback of $\omega_{(H)}$ by the orbit map $J^{-1}(\mathcal{O}) \cap M_{(H)} \to (M_{\mathcal{O}})_{(H)}$ coincides with the restriction to $J^{-1}(\mathcal{O}) \cap M_{(H)}$ of the symplectic form ω. Finally, the decomposition of $M_{\mathcal{O}} := J^{-1}(\mathcal{O})/G$, the reduced space of M at the orbit $\mathcal{O}$, given by

$$M_{\mathcal{O}} = \coprod_{H<G} (M_{\mathcal{O}})_{(H)}$$

is a symplectic stratification of $M_{\mathcal{O}}$.

It is a curious fact that each stratum $(M_{\mathcal{O}})_{(H)}$ may also obtained by a regular Marsden-Weinstein reduction. To keep the discussion simple let us assume that $\mathcal{O}$ is the zero orbit. (This is no loss of generality by virtue of the shifting trick.) For a subgroup H of G define

$$M_H = \{\, m \in M : G_m \text{ is exactly } H \,\}$$

It is well-known that M_H is a symplectic submanifold of M. The action of G does not preserve the manifold M_H. However, the smaller group $L = N_G(H)/H$ does act on M_H, where $N_G(H)$ denotes the normalizer of H in G. Moreover, the action of L is Hamiltonian and the corresponding moment map $J_L : M_H \to \mathfrak{l}^*$ is essentially the restriction of the moment map $J : M \to \mathfrak{g}^*$ to M_H.

2.22. Theorem (cf. [21]). *Zero is a regular value of the moment map J_L. The Marsden-Weinstein reduced space $(J_L)^{-1}(0)/L$ is symplectically isomorphic to the stratum $(M_0)_{(H)}$.*

This theorem provides us with a simple recipe for lifting integral curves of a reduced Hamiltonian flow on the reduced space M_0 to the level set $J^{-1}(0)$. Namely, let $\bar{h}$ be an invariant smooth function on the manifold M, and let h be the smooth function on the reduced space induced by $\bar{h}$. Let $\bar{\Phi}_t$ and Φ_t, denote the Hamiltonian flows of $\bar{h}$ and h, respectively. If $\gamma(t)$ is an integral curve of the function h, then it lies inside some stratum $(M_0)_{(H)}$, and the classical recipe for lifting a reduced flow (see e.g. [1]) can be used to lift $\gamma(t)$ to an integral curve of the Hamiltonian $\bar{h}$, lying in the manifold M_H.

2.4 The Closure of a Coadjoint Orbit as a Stratified Symplectic Space

The object of this section is to show that for a large class of Lie groups the closure of every coadjoint orbit is a stratified symplectic space. In Section 4 we shall see that in some cases a reduced space of a Hamiltonian space can be identified with the closure of a coadjoint orbit of a different group.

2.23. Theorem. *Let H be a reductive Lie group and let $\mathcal{O} \subset \mathfrak{h}^*$ be a coadjoint orbit of H. Then the closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ is a stratified symplectic space. The strata are the H-orbits in $\bar{\mathcal{O}}$.*

Proof. We take the space $C^\infty(\bar{\mathcal{O}})$ of smooth functions on $\bar{\mathcal{O}}$ to be the space of *Whitney smooth* functions. Recall that a continuous map $f : \bar{\mathcal{O}} \to \mathbf{R}$ is called Whitney smooth if and only if there exists a function $F \in C^\infty(\mathfrak{h}^*)$ such that $F|_{\bar{\mathcal{O}}} = f$.

It is easy to see that $C^\infty(\bar{\mathcal{O}})$ is naturally a Poisson algebra. Indeed, since the coadjoint orbits are the symplectic leaves of the Poisson structure on $\mathfrak{h}^*$, for all F, $G \in C^\infty(\mathfrak{h}^*)$ and $x \in \bar{\mathcal{O}}$ the bracket $\{F, G\}(x)$ depends only on the restrictions of F and G to the coadjoint orbit of x, which is contained in $\bar{\mathcal{O}}$. Thus the bracket $\{\cdot,\cdot\}_{\bar{\mathcal{O}}}$ given by

$$\{F|_{\bar{\mathcal{O}}}, G|_{\bar{\mathcal{O}}}\}_{\bar{\mathcal{O}}}(x) := \{F, G\}(x)$$

is well-defined. The partition of $\bar{\mathcal{O}}$ into coadjoint orbits is a decomposition. The local finiteness follows from the assumption that H is reductive. The proof of the fact that $\bar{\mathcal{O}}$ is a *stratified* space requires some machinery.

2.24. Definition. Let X be a subspace of $\mathbf{R}^n$. A decomposition of X is called a *Whitney stratification* if the pieces of X are smooth submanifolds of $\mathbf{R}^n$ and if for each pair of pieces P, Q with $P \leq Q$ the following condition of Whitney holds:

Whitney's Condition B. Let p be an arbitrary point in P and let $\{p_i\}$ and $\{q_i\}$ be sequences in P, resp. Q, both converging to p. Assume that the lines l_i joining p_i and q_i converge (in the projective space $\mathbf{R}P^{n-1}$) to a line l, and that the tangent planes $T_{q_i}Q$ converge (in the Grassmannian of $(\dim Q)$-planes in $\mathbf{R}^n$) to a plane τ. Then l is contained in τ.

It follows from Mather's theory of control data (see [14]) that a Whitney stratified subset of Euclidean space is a stratified space in the sense of our Definition 2.6. An outline of the argument can be found in [8, page 40]. So it suffices to show that $\bar{\mathcal{O}}$ is a Whitney stratified space.

Since H is reductive, the coadjoint representation $Ad^* : H \to Gl(\mathfrak{h}^*)$ is algebraic, i.e., the image $Ad^*(H)$ is an algebraic subgroup of $Gl(\mathfrak{h}^*)$ and the coadjoint action of $Ad^*(H)$ on $\mathfrak{h}^*$ is algebraic (see e.g. [23] for a proof). Now a coadjoint orbit $Ad^*(H)\cdot q$, $q \in \mathfrak{h}^*$, is semialgebraic by the Seidenberg-Tarski theorem, since it is the image of $Ad^*(H)$ under the algebraic map 'evaluation at q', which sends $a \in Ad^*(H)$ to $a \cdot q$. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two orbits in $\bar{\mathcal{O}}$ with $\mathcal{O}_1$ contained in the closure of $\mathcal{O}_2$. The two orbits are smooth and semialgebraic. Therefore a theorem of Wall [24, p. 337] applies. In this case the theorem says that Whitney's condition B for the pair $(\mathcal{O}_1, \mathcal{O}_2)$ holds at all points of $\mathcal{O}_1$ except possibly for the points in a semialgebraic subvariety of dimension strictly less than the dimension of $\mathcal{O}_1$. In particular condition B holds at *some* point of $\mathcal{O}_1$. But the pair $(\mathcal{O}_1, \mathcal{O}_2)$ is H-homogeneous, so condition B holds everywhere. This proves that $\bar{\mathcal{O}}$ is a Whitney stratified space. □

3 Reduction of Cotangent Bundles

3.1 The Cotangent Bundle of a Quotient Variety

We have seen in Section 1 that the singular reduced space $(T^*\mathbf{R}^2)_0$ is a symplectic orbifold. There are a few other interesting examples of singular reduced spaces coming from reduction of cotangent bundles which turn out to be orbifolds. In order to understand what makes these examples work it will be helpful to consider lifted actions on cotangent bundles in general. (We caution the reader that not every reduced space is an orbifold; see [5] for a counterexample.) Let G be a Lie group acting smoothly and properly on a smooth manifold X. Let x be a point in X and H the stabilizer of x in G. Since the action is proper H is compact. Therefore there exists an H-equivariant splitting of the tangent space to X at x:

$$T_xX = T_x(G \cdot x) \oplus V$$

where V is some subspace of the tangent space. Let B be a small H-invariant ball in V centered at the origin. The slice theorem asserts that a neighbourhood of the orbit $G\cdot x$ in X is G-equivariantly diffeomorphic to the associated bundle $G\times_H B$.

If the action of G is free, it follows from the slice theorem that X is a principal G-bundle over the orbit space $Q = X/G$. Lift the action of G to an action on the cotangent bundle. It is well-known (see e.g. [1]) that in this case the reduced space at the zero level is simply the cotangent bundle of the base, $(T^*X)_0 = T^*Q$. This result has been recently generalized by Emmrich and Römer [6] to the case when the action of G on X is of constant orbit type, that is, there exists a subgroup H of G such that for any $x \in X$ the orbit $G \cdot x$ is diffeomorphic to the homogeneous space G/H. Alternatively, by virtue of the slice theorem, the action of G on X is of constant orbit type if and only if X is a fibre bundle over the orbit space $Q = X/G$ with typical fibre G/H. Emmrich and Römer showed that in this case the reduced space $(T^*X)_0$ is again T^*Q, the cotangent bundle of the orbit space.

Let us now consider the general case of an action of G on X, that is, we make no assumption concerning the structure of the orbits. Lift the action of G to an action on the cotangent bundle T^*X and let $J : T^*X \to \mathfrak{g}^*$ be the corresponding moment map. Recall that for $(x,\eta) \in T^*_xX$ the value of J is defined by

$$\langle \xi, J(x,\eta)\rangle = -\langle \xi_X(x), \eta\rangle, \tag{7}$$

where $\langle\cdot,\cdot\rangle$ on the left hand side of the equation denotes the pairing between the Lie algebra $\mathfrak{g}$ and its dual, and on the right hand side the pairing between the tangent and the cotangent spaces of X at x, while $\xi_X(x)$ is the vector obtained by evaluating at x the vector field defined by the infinitesimal action ξ on X. Let us compute the zero level set of the moment map. It follows from (7) that

$$J^{-1}(0)\cap T^*_xX = \{\,\eta : \langle \xi_X(x), \eta\rangle = 0 \text{ for all } \xi \in \mathfrak{g}\,\}.$$

We have proved:

3.1. Lemma. *Let $J : T^*X \to \mathfrak{g}^*$ be the moment map induced by the lift of the action G on X to an action on T^*X. Then the intersection of the zero level set of the moment map with the fibre of the cotangent space at a point $x \in X$ is $(T_x(G\cdot x))^\circ$, the annihilator of the tangent space to the orbit through x. Consequently,*

$$J^{-1}(0) = \coprod_{x\in X} (T_x(G\cdot x))^\circ. \tag{8}$$

3.2. Remark. It follows from the description (8) of the zero level set that it retracts onto X. In particular, if X is connected then the level set $J^{-1}(0)$ is connected as well.

For a point x in X we call the orbit space $(T_x(G\cdot x))^\circ/G_x$ the *cotangent cone* of X/G at the point $G\cdot x \in X/G$. It is easy to see that this definition does not depend on the choice of the point $x \in G\cdot x$, i.e., if $x' = a\cdot x$ for some $a \in G$, then multiplication by a induces an isomorphism between the orbit spaces $(T_x(G\cdot x))^\circ/G_x$ and $(T_{x'}(G\cdot x'))^\circ/G_{x'}$. Moreover, the quotient

$$J^{-1}(0)/G = \Big(\coprod_{x\in X} (T_x(G\cdot x))^\circ\Big)\Big/G$$

is set-theoretically the disjoint union of all cotangent cones to X/G. Therefore the following definition makes sense.

3.3. Definition. The *cotangent bundle* of an orbit space X/G is the stratified symplectic space $T^*(X/G) := J^{-1}(0)/G$.

3.4. Example. Suppose that G is finite. Then $J = 0$, so $T^*(X/G) := J^{-1}(0)/G = T^*(X)/G$.

3.5. Remark. The cotangent bundle $T^*(X/G)$ is not a locally trivial bundle over the base variety X/G, since the fibres may vary from point to point. Nor is the projection $T^*(X/G)\to X/G$ a stratification-preserving map.

3.6. Example. Let $X = \mathbf{R}^2$ and let $G = SO(2)$ act on X in the standard way. Then the quotient X/G is a closed half-line $[0,\infty)$. It consists of two strata: the end-point $\{0\}$ and the open half-line $(0,\infty)$. We saw in Section 1 that the cotangent bundle $T^*(X/G)$ of the half-line is a cone. The fibre $\pi^{-1}(x)$ of the projection $\pi : T^*(X/G)\to X/G$ is a line if $x\in(0,\infty)$, but it is a closed half-line if $x = 0$. So $T^*(X/G)$ is not a locally trivial bundle over X/G. Notice that $\pi^{-1}(0)$ intersects the top stratum of $T^*(X/G)$. So the preimage of the stratum $\{0\}$ is not a union of strata.

It seems unlikely to us that the smooth structure of a cotangent bundle $T^*(X/G)$ depends on the way in which the orbit space X/G is written as a quotient. More precisely, we make the following

3.7. Conjecture. *Let G and H be Lie groups and let X, resp. Y, be smooth manifolds on which G, resp. H act properly. Assume that the orbit spaces X/G and Y/H are diffeomorphic in the sense that there exists a homeomorphism $\phi : X/G \to Y/H$ such that the pullback map ϕ^* is an isomorphism from $C^\infty(Y/H) := C^\infty(Y)^H$ to $C^\infty(X/G) := C^\infty(X)^G$. Then the cotangent bundles of X/G and Y/H are isomorphic in the sense of Definition 2.14.*

In his unpublished thesis [19], Schwarz showed that modulo some assumptions $T^*(X/G)$ and $T^*(Y/H)$ are *homeomorphic* if X/G and Y/H are diffeomorphic. In the next sections we prove a version of this result and provide some experimental evidence for Conjecture 3.7.

3.2 Cross-sections

Let X be a smooth manifold and G a Lie group acting on X. Often one can compute the cotangent bundle of the quotient variety X/G by means of a *cross-section* of the G-action, i.e., a pair (Y,H), where Y is an embedded submanifold of X and H a Lie group acting on Y such that every G-orbit in X intersects Y in exactly one H-orbit. If (Y,H) is a cross-section, it is easy to see that the natural map $Y/H \to X/G$ is a homeomorphism. On an additional assumption we show now that the cotangent bundles $T^*(X/G)$ and $T^*(Y/H)$ are also homeomorphic.

3.8. PROPOSITION. *Let X be a Riemannian G-manifold. Assume that (Y,H) is a cross-section of the G-action on X. Assume further that the cross-section is orthogonal in the sense that for all y in Y*

$$T_y(G\cdot y) = \big((T_yY)^\perp \cap (T_y G\cdot y)\big) \oplus \big(T_yY \cap T_y(G\cdot y)\big). \tag{9}$$

Then the inclusion $Y \subset X$ induces a homeomorphism $(T^*Y)_0 \xrightarrow{\simeq} (T^*X)_0$.

3.9. REMARK. Suppose the cross-section Y is the set of fixed points for some subgroup K of G. Let H be the 'Weyl group' $N(K)/K$. The statement (9) regarding the orthogonality of the intersections of the G-orbits with Y holds automatically in this case. This follows easily from the proof of the slice theorem.

PROOF. The metric allows us to identify equivariantly tangent and cotangent bundles of X and of Y, giving rise to a symplectic embedding

$$T^*Y \simeq TY \hookrightarrow TX \simeq T^*X.$$

Let $J_X : TX \to \mathfrak{g}^*$ and $J_Y : TY \to \mathfrak{h}^*$ denote the moment maps. Let y be a point in Y. Since the orbit $G\cdot y$ intersects Y in a single H-orbit, (9) implies that

$$T_y(G\cdot y) = \big((T_yY)^\perp \cap (T_y G\cdot y)\big) \oplus (T_y(H\cdot y))$$

is an orthogonal decomposition. Hence $(T_y(G\cdot y))^\perp = V_1 \oplus V_2$, where $V_1 = (T_y H\cdot y)^\perp \cap T_yY$ and $V_2 = (T_y G\cdot y)^\perp \cap (T_yY)^\perp$, so that

$$J_X^{-1}(0) \cap T_y^*X = V_1 \oplus V_2$$

and

$$J_Y^{-1}(0) \cap T_y^*Y = V_1.$$

This gives us an inclusion $J_Y^{-1}(0) \to J_X^{-1}(0)$. Composing with the orbit map $J_X^{-1}(0) \to J_X^{-1}(0)/G = (T^*X)_0$ gives us a map from $J_Y^{-1}(0)$ to $(T^*X)_0$. We

claim that this map descends to a map from $J_Y^{-1}(0)/H$ to $(T^*X)_0$. Indeed, suppose (y',η') and (y,η) are two points in $J_Y^{-1}(0)$ and $a\cdot(y',\eta') = (y,\eta)$ for some $a \in H$. By assumption $G\cdot y \cap Y = H\cdot y$, so there is $b \in G$ with $b\cdot y' = y$. It is therefore no loss of generality to assume that $y = y'$. In this case η and η' both lie in V_1 and $a \in H_y$. Locally near y the space Y is H-equivariantly diffeomorphic to the associated bundle $H \times_{H_y} V_1$, so locally

$$Y/H \simeq (H \times_{H_y} V_1)/H = V_1/H_y.$$

Here H_y denotes the stabilizer of y in H. Similarly,

$$X/G \simeq (G \times_{G_y} (V_1 \oplus V_2))/G = (V_1 \oplus V_2)/G_y,$$

where G_y denotes the stabilizer of y in G. We have assumed that (Y,H) is a cross-section for the G-action, and therefore $X/G \simeq Y/H$. It follows that $\eta,\eta' \in V_1$ lie in the same H_y-orbit if and only if $\eta,\eta' \in V = V_1 \oplus V_2$ lie in the same G_y orbit. We conclude that there is $c \in G$ with $c\cdot(y,\eta) = (y',\eta')$, thereby proving the existence of a continuous map

$$\varphi : (T^*Y)_0 = J_Y^{-1}(0)/H \to (T^*X)_0.$$

A similar argument shows that φ is bijective and that φ^{-1} is continuous. □

3.10. REMARK. This proof shows that each G-orbit in $J_X^{-1}(0)$ intersects $J_Y^{-1}(0)$ in a single H-orbit, in other words, that the pair $(J_Y^{-1}(0), H)$ is a cross-section of the G-action on $J_X^{-1}(0)$.

3.3 Row, Row, Row your Boat

Let X be the unit two-sphere in $\mathbf{R}^3$ and let G be the circle acting on X by rotations on the z-axis. The space X is the configuration space of the spherical pendulum and G is its group of symmetries. Now let Y be a great circle through the poles and let H be the group $\mathbf{Z}_2$ acting on Y by reflection in the z-axis. Then the pair (Y,H) is obviously an orthogonal cross-section of the G-action on X. Let J_X be the momentum map of the lifted action of G on T^*X. The lifted action of H on T^*Y has trivial momentum map, since H is finite. By Proposition 3.8, the pair (T^*Y, H) is a cross-section for the G-action on $J_X^{-1}(0)$. The physical meaning of this fact is that a spherical pendulum with zero angular momentum is just a planar pendulum.

Let us describe the orbifold $(T^*Y)/H$ in some detail. We identify the meridian Y with $S^1 = \{\,e^{i\theta} : \theta \in \mathbf{R}\,\}$ in such a manner that the south pole is mapped to $1 \in S^1$. We cover $(T^*Y)/H = (S^1 \times \mathbf{R})/\mathbf{Z}_2$ with two orbifold charts. The domain of both charts is the strip $D = (-\pi,\pi) \times \mathbf{R} \subset \mathbf{R}^2$

equipped with the $\mathbf{Z}_2$-action generated by reflection in the origin. The chart maps ψ_1 and ψ_2 are given by:

$$\psi_1 : D \to (S^1 \times \mathbf{R})/\mathbf{Z}_2, \quad (\theta, r) \mapsto [e^{i\theta}, r],$$
$$\psi_2 : D \to (S^1 \times \mathbf{R})/\mathbf{Z}_2, \quad (\theta, r) \mapsto [-e^{i\theta}, r],$$

where $[x,y]$ denotes the equivalence class of $(x,y) \in S^1 \times \mathbf{R}$. It is easy to write down the transition map from one chart to the other. The resulting space has the shape of a 'canoe' with two isolated conical singularities. We encourage the reader to construct this orbifold with paper and glue.

We claim that the natural homeomorphism

$$\phi : (T^*Y)/H = T^*(Y/H) \to T^*(X/G) \tag{10}$$

is an isomorphism of reduced spaces. It obviously suffices to show that $\phi : O_i \to \phi(O_i)$ is an isomorphism, where $O_i = \psi_i(D)$ for $i = 1, 2$. Note that $\phi(\psi_i(D))$ is the space obtained by reducing $T^*(X - \{*\})$ at zero, where $\{*\}$ is either the south or the north pole of the sphere X, depending on whether $i = 1$ or 2. But $X - \{*\}$ is G-equivariantly diffeomorphic to the plane $\mathbf{R}^2$, if we let $G = SO(2)$ act on $\mathbf{R}^2$ in the standard fashion. So the maps $\phi : O_i \to \phi(O_i)$ are, up to changes of coordinates, equal to the map $\mathbf{R}^2/\mathbf{Z}_2 \to (T^*\mathbf{R}^2)_0$ exhibited in Section 1, which is an isomorphism. Therefore, the map (10) is also an isomorphism.

The two isolated singularities of the 'canoe' are relative equilibria of the spherical pendulum. Both are actually absolute equilibria, corresponding to the pendulum pointing straight up or down. For an alternative computation of the 'canoe' using invariant polynomials, see [3].

3.4 Reduction of the Cotangent Bundle of a Symmetric Space

Consider the special orthogonal group $SO(n)$ acting by conjugation on $S^2(\mathbf{R}^n)$ the space of real symmetric $n \times n$-matrices. Let S_n denote the symmetric group on n letters acting on $\mathbf{R}^n$ by permuting the coordinates and hence on $T^*\mathbf{R}^n$ by permuting the coordinates in pairs. Note that $\mathbf{R}^n$ embeds into $S^2(\mathbf{R}^n)$ as the set of diagonal matrices. Since any symmetric matrix is diagonalizable, the pair $(\mathbf{R}^n, S_n)$ is a cross-section of the $SO(n)$-action on $S^2(\mathbf{R}^n)$. Therefore $S^2(\mathbf{R}^n)/SO(n)$ is homeomorphic to $\mathbf{R}^n/S_n$. The vector space $S^2(\mathbf{R}^n)$ has a natural $SO(n)$-invariant inner product:

$$((a_{ij}), (b_{kl})) = \mathrm{trace}((a_{ij})(b_{kl})) = \sum_{ij} a_{ij} b_{ij}.$$

Remark 3.9 implies that the cross-section $(\mathbf{R}^n, S_n)$ is orthogonal. Therefore Proposition 3.8 provides us with a homeomorphism $\phi : (T^*\mathbf{R}^n)_0 \to (T^*S^2(\mathbf{R}^n))_0$. We contend that ϕ is an isomorphism of reduced spaces. Since the group S_n is finite, the zero level set of the S_n-moment map is the whole space $\mathbf{R}^n \times \mathbf{R}^n$, which embeds naturally into $T^*S^2(\mathbf{R}^n) \simeq S^2(\mathbf{R}^n) \times S^2(\mathbf{R}^n)$. In fact $\mathbf{R}^n \times \mathbf{R}^n$ is a subset of the zero level set of the $SO(n)$-moment map $J : T^*S^2(\mathbf{R}^n) \to \mathfrak{so}(n)^*$. Clearly any $SO(n)$-invariant function on $T^*S^2(\mathbf{R}^n)$ restricts to an S_n-invariant function on $\mathbf{R}^n \times \mathbf{R}^n$. This implies that $\phi^*C^\infty((T^*S^2(\mathbf{R}^n))_0)$ is contained in $C^\infty((T^*\mathbf{R}^n)_0) = C^\infty(\mathbf{R}^n \times \mathbf{R}^n)^{S_n}$. To show that ϕ is an isomorphism of reduced spaces we need to prove that $\phi^*C^\infty(T^*S^2(\mathbf{R}^n))$ is equal to $C^\infty((T^*\mathbf{R}^n)_0)$. By the same argument as the one we have used in the example of Section 1, it is enough to show that there is a set $\{\sigma_{ij}\}$ of polynomials that generates the S_n-invariant polynomials on $\mathbf{R}^n \times \mathbf{R}^n$ and has the property that each σ_{ij} is the restriction of an $SO(n)$-invariant polynomial on $S^2(\mathbf{R}^n) \times S^2(\mathbf{R}^n)$. According to Weyl [25], the polynomials

$$\sigma_{kl}(x,y) = \sum_{ij} x_i^k y_j^l, \quad 1 \le k, l \le n, \tag{11}$$

generate the S_n-invariant polynomials on $\mathbf{R}^n \times \mathbf{R}^n$. On the other hand, σ_{kl} is the restriction of the $SO(n)$-invariant polynomial $\tau_{kl}(A,B) = \mathrm{trace}(A^k B^l)$, so the polynomials (11) are the required set. We have thus proved that

$$(T^*S^2(\mathbf{R}^n))_0 \simeq (\mathbf{R}^n \times \mathbf{R}^n)/S_n$$

as stratified symplectic spaces.

More generally, let G be a semisimple Lie group over $\mathbf{R}$, K a maximal compact subgroup of G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g} = Lie(G)$. Then K acts on $\mathfrak{p}$ by conjugation. Pick a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let $W = N(\mathfrak{a})/C(\mathfrak{a})$ denote the Weyl group. It is well-known that the restriction map $\mathbf{R}[\mathfrak{p}] \to \mathbf{R}[\mathfrak{a}]$ from polynomials on $\mathfrak{p}$ to polynomials on $\mathfrak{a}$ gives rise to an isomorphism $\mathbf{R}[\mathfrak{p}]^K \to \mathbf{R}[\mathfrak{a}]^W$. The quotient spaces $\mathfrak{a}/W$ (the Weyl chamber) and $\mathfrak{p}/K$ are therefore isomorphic. The computation above verifies Conjecture 3.7 in the special case $G = Sl(n, \mathbf{R})$, showing that we have an isomorphism of cotangent bundles, $(T^*\mathfrak{a})/W = T^*(\mathfrak{a}/W) \cong T^*(\mathfrak{p}/K)$. For arbitrary G, Conjecture 3.7 would follow from (but is not equivalent to):

3.11. CONJECTURE. *The restriction map* $\mathbf{R}[\mathfrak{p} \times \mathfrak{p}]^K \to \mathbf{R}[\mathfrak{a} \times \mathfrak{a}]^W$ *is surjective.*

4 Poisson Embeddings of Reduced Spaces

The goal of this section is to show that in some cases a reduced space of a symplectic representation space can be realized as the closure of a coadjoint orbit in the dual of some Lie algebra (cf. Section 2.4). For the remainder of this section, let K be a compact group acting linearly on a symplectic vector space V and preserving its symplectic form ω. Then the action of K is

Hamiltonian. Let $J : V \to \mathfrak{k}^*$ denotes the corresponding moment map. The ring of invariant polynomials $\mathbf{R}[V]^K$ is finitely generated. We now make the following assumption:

ASSUMPTION Q. The ring of all K-invariant polynomials on V is generated by the homogeneous quadratic K-invariant polynomials.

The space of homogeneous quadratic polynomials, $\mathbf{R}_2[V]$, and the space of invariant polynomials are both closed under the Poisson bracket. It follows that their intersection,

$$\mathfrak{h} := \mathbf{R}_2[V]^K,$$

which is the space of invariant homogeneous quadratic polynomials, is also closed under the Poisson bracket. The algebra $\mathbf{R}_2[V]$ is canonically isomorphic to the Lie algebra $\mathfrak{sp}(V)$ of all infinitesimally symplectic linear transformations: the isomorphism takes a quadratic polynomial to its associated Hamiltonian vector field. The inverse map sends $\xi \in \mathfrak{sp}(V,\omega)$ to the polynomial $1/2\,\omega(\xi v, v)$. Thus we can view $\mathfrak{h}$ as a subalgebra of $\mathfrak{sp}(V)$.

Consider the map $\sigma : V \to \mathfrak{h}^*$ defined by

$$\langle \sigma(v), P\rangle = P(v)$$

where $P \in \mathfrak{h}$ and $\langle \cdot,\cdot\rangle$ denotes the canonical pairing of a vector space with its dual. This is the Hilbert map of classical invariant theory. It is manifestly K-invariant, and so induces a map $\bar{\sigma} : V/K \to \mathfrak{h}^*$. Assumption Q above implies that σ separates K-orbits. Thus $\bar{\sigma}$ is a homeomorphism onto its image $\sigma(V) \subset \mathfrak{h}^*$. Let H be the connected subgroup of $Sp(V)$ whose Lie algebra is $\mathfrak{h}$. Note that the map σ is the momentum map for the H action on V. It is H-equivariant. (Here H acts on $\mathfrak{h}^*$ by the coadjoint action.)

4.1. REMARK. It is perhaps helpful to rephrase the above discussion in coordinates. Let $\sigma_1, \ldots, \sigma_N$ be a basis for the space $\mathfrak{h}$ of invariant homogeneous quadratic polynomials. The Poisson bracket of any two generators is again a homogeneous quadratic K-invariant polynomial (or zero), which demonstrates that $\mathfrak{h}$ is a Lie algebra. This Lie algebra has structure constants $c_{ij}^k \in \mathbf{R}$ defined by

$$\{\sigma_i, \sigma_j\} = \sum_k c_{ij}^k \sigma_k.$$

The map σ, in terms of this choice of coordinates on $\mathfrak{h}^*$ is

$$\sigma : V \to \mathbf{R}^N, \quad v \mapsto (\sigma_1(v), \ldots, \sigma_N(v)),$$

where we have identified $\mathbf{R}^N$ with $\mathfrak{h}^*$, the isomorphism being the one associated to choosing the basis of $\mathfrak{h}^*$ which is dual to the basis σ_i.

4.2. REMARK. Motivated by problems in representation theory, Howe [9] defined a *reductive dual pair* to be a pair of reductive subgroups of $Sp(V)$ that are each other's centralizers. The groups K and H above clearly commute with each other and it is easy to see that H is (the identity component of) the centralizer of K. It is not true in general that K is the centralizer of H, as the example of $K = SU(2)$ acting on $V = \mathbf{C}^2$ clearly indicates. One can get around the problem of K not being the full centralizer of H in $Sp(V)$ by replacing it with $K' :=$ the centralizer of H in $Sp(V)$. However, it is not at all clear why K' and H should be reductive. Also, given a dual pair (K, H) with K compact, it is not clear whether the quadratic polynomials corresponding to $\mathfrak{h} = Lie(H)$ generate $\mathbf{R}[V]^K$.

However, in three interesting physical examples of symplectic representations of K satisfying condition Q, the groups K and H do form a reductive dual pair:

1. the planar N-body problem ($SO(2)$ acting diagonally on $(T^*\mathbf{R}^2)^N$);
2. the d-dimensional N-body problem ($O(d)$ acting diagonally on $(T^*\mathbf{R}^d)^N$), this example is worked out in the next section;
3. $U(p)$ acting on $\mathbf{C}^p \otimes \mathbf{C}^q$.

These examples seem to hint at an interesting connection between reductive dual pairs and condition Q.

Now let $\mathcal{O}$ be a coadjoint orbit of K. Consider the corresponding reduced space $V_{\mathcal{O}} = J^{-1}(\mathcal{O})/K$. We claim that the map $\bar{\sigma} : V_{\mathcal{O}} \to \mathfrak{h}^*$ induced by the H-momentum map σ is a Poisson embedding in the following sense.

4.3. DEFINITION. Let X be a stratified symplectic space and let P be a Poisson manifold. A *proper Poisson embedding* of X into P is a proper injective map $j : X \to P$ such that

i. the pullback by j of every smooth function on P is a smooth function on X;
ii. the pullback map $j^* : C^\infty(P) \to C^\infty(X)$ is surjective;
iii. the pullback map j^* is a morphism of Poisson algebras.

We mention a few obvious consequences of this definition: the image of a proper Poisson embedding $j : X \to P$ is closed; j is a homeomorphism onto its image; the kernel of j^*, which is the set of smooth functions vanishing on the image $j(X)$, is a Poisson ideal inside $C^\infty(P)$; and the set of Whitney smooth functions on $j(X)$ is a Poisson algebra, which is isomorphic to $C^\infty(X)$. Therefore $j(X)$ is a stratified symplectic space (stratified by the images of the strata in X) and $j : X \to j(X)$ is an isomorphism of stratified symplectic spaces.

4.4. THEOREM. *Suppose that assumption Q holds. Let H be the closed connected Lie subgroup of $Sp(V)$ described above, and $\sigma : V \to \mathfrak{h}^*$ its associated momentum map. Let $\mathcal{O}$ be an arbitrary coadjoint orbit of K. Then the following statements hold.*

1. *The map $\bar{\sigma} : V_{\mathcal{O}} \to \mathfrak{h}^*$ is a proper Poisson embedding of the K-reduced space $V_{\mathcal{O}}$ (where the bracket on $\mathfrak{h}^*$ is the usual Lie-Poisson bracket);*
2. *Each connected component of a symplectic stratum of $V_{\mathcal{O}}$ is mapped symplectomorphically by $\bar{\sigma}$ onto a coadjoint orbit of H contained in $\bar{\sigma}(V_{\mathcal{O}})$;*
3. *The image $\bar{\sigma}(V_{\mathcal{O}})$ of the Poisson embedding is the closure of a single coadjoint orbit of H.*

PROOF. 1. We check the conditions of Definition 4.3. The square of the distance to the origin in V is a K-invariant polynomial function. From this it follows easily that the Hilbert map σ is proper. Hence the map $\bar{\sigma} : V_{\mathcal{O}} \to \mathfrak{h}^*$ is proper. It is injective because the Hilbert map separates the K-orbits. It is not hard to see from the definition of smooth functions on $V_{\mathcal{O}}$ that $\bar{\sigma}$ pulls back smooth functions to smooth functions. That the pullback map $\bar{\sigma}^* : C^\infty(\mathfrak{h}^*) \to C^\infty(V_{\mathcal{O}})$ is surjective is an easy consequence of Schwarz's theorem [20]. It is a homomorphism of Poisson algebras, because the Hilbert map σ, being the H-momentum map, is a Poisson map.
2. The connected components of the symplectic strata are the symplectic leaves of the reduced space $V_{\mathcal{O}}$, i.e., they are swept out by the Hamiltonian flows of smooth functions (see Lemma 2.19). Since the Poisson algebras $C^\infty(V_{\mathcal{O}})$ and $C^\infty(j(V_{\mathcal{O}}))$ are isomorphic, the embedding j maps leaves onto leaves. But the leaves of $\mathfrak{h}^*$ are simply the coadjoint H-orbits. (Here we use that H is connected.)
3. Theorem 4.6 below states that the level set $J^{-1}(\mathcal{O})$ is connected. It follows now from Theorem 2.12 that the reduced space $V_{\mathcal{O}}$ has a connected open dense stratum S_{top}; so the set $\bar{\sigma}(V_{\mathcal{O}})$ has to be the closure of $\bar{\sigma}(S_{top})$, which is a single coadjoint orbit by statement 2 of this theorem.

□

4.5. REMARK. Denote the stratified symplectic space $\bar{\sigma}(V_{\mathcal{O}})$ by $X_{\mathcal{O}}$. If the group H is semisimple then we use a Killing form to identify $\mathfrak{h}$ with $\mathfrak{h}^*$ in an H-equivariant way. If $\mathcal{O}$ is the zero orbit, then the image $X_0 = \sigma(J^{-1}(0))$ described in Theorem 4.4 is neccessarily the closure of a *nilpotent* orbit. This is because $J(0) = 0$ and $\sigma(0) = 0$ so that $0 \in \sigma(J^{-1}(0))$. And the only orbits whose closure contains 0 are the nilpotent ones.

More generally $X_{\mathcal{O}}$ contains a single semisimple orbit orbit Q and any other orbit P contained in $X_{\mathcal{O}}$ fibres over Q. The fibration $\pi_P : P \to Q$ is simply the projection of $\eta \in P$ onto its semisimple part, $\pi_P(\eta) = \eta_{ss}$. The fibre of π_P is the orbit of the nilpotent part η_n of η under the action of the stabilizer group $H_{\eta_{ss}}$ of η_{ss}. Note that η_n is nilpotent in $Lie(H_{\eta_{ss}})$. It follows that one can view the map $\pi : X_{\mathcal{O}} \to Q$ as a fibre bundle with typical fibre being the closure of a nilpotent orbit in some smaller reductive group. These facts about the structure of orbits of a semisimple group are well-known and we refer the reader to [23] for proofs and further references. It was shown in [12] that if a (co)adjoint orbit P fibres over an semisimple orbit Q then the fibration is symplectic. Thus the map $\pi : X_{\mathcal{O}} \to Q$ can be viewed as a fibration of stratified symplectic spaces.

To conclude this section, we prove the connectivity statement used in the proof of Theorem 4.4. This result does not use assumption Q.

4.6. THEOREM. *Let K be a compact group acting linearly on a symplectic vector space V and preserving its symplectic form ω. Let $J : V \to \mathfrak{k}^*$ denotes the corresponding moment map. Then for any coadjoint orbit $\mathcal{O}$ of K the set $J^{-1}(\mathcal{O})$ is connected.*

PROOF. Without loss of generality we may assume that V is $\mathbf{C}^n$ with the standard symplectic form and K is a subgroup of the unitary group $U(n)$. Let $\mathcal{O}$ be a coadjoint orbit of K. We will show that for any $r > 0$ the closed ball

$$\bar{B}(r) = \{\, z \in \mathbf{C}^n : |z|^2 \leq r \,\}$$

intersects $J^{-1}(\mathcal{O})$ in a connected set. Clearly this will prove the theorem.

Note first that the central circle subgroup of $U(n)$,

$$U(1) = \left\{ \begin{pmatrix} e^{i\theta} & & \\ & \ddots & \\ & & e^{i\theta} \end{pmatrix} : \theta \in \mathbf{R} \right\}$$

commutes with K and therefore preserves the level set $J^{-1}(\mathcal{O})$. Consider now the space $N(r)$ obtained from $\bar{B}(r)$ by identifying the points on the boundary that lie in the same $U(1)$-orbit. Let $q : \bar{B}(r) \to N(r)$ denote the quotient map. Since $J^{-1}(\mathcal{O})$ is $U(1)$-invariant and the fibres of q are connected, the set $J^{-1}(\mathcal{O}) \cap \bar{B}(r)$ is connected if and only if its image under q is connected in $N(r)$. We will see shortly that $N(r)$ is K-equivariantly symplectomorphic to $\mathbf{C}P^n(r)$, the complex projective space with the symplectic form equal to the standard one times r. We will also see that under this identification the action of K on $N(r)$ becomes Hamiltonian with the moment map $J_r : N(r) \to \mathfrak{k}^*$ having the property that $J_r^{-1}(\mathcal{O}) = q(J^{-1}(\mathcal{O}) \cap \bar{B}(r))$.

Consider the action of $U(1)$ on $\mathbf{C}^n \times \mathbf{C}$ corresponding to the Hamiltonian $\phi(z, w) = |z|^2 + |w|^2 - r$ for $(z, w) \in \mathbf{C}^n \times \mathbf{C}$. Then

$$\phi^{-1}(0) = \{ (z, w) \in \mathbf{C}^n \times \mathbf{C} : |z|^2 + |w|^2 = r \}$$

and $\phi^{-1}(0)/U(1) \simeq \mathbf{C}P^n(r)$. Now, K acts on $\mathbf{C}^n \times \mathbf{C}$ by acting trivally on the second factor. Since the actions of K and $U(1)$ on $\mathbf{C}^n \times \mathbf{C}$ commute, the action of K descends to a Hamiltonian action on the reduced space $\mathbf{C}P^n(r)$. The corresponding moment map J_r is obtained by extending $J : \mathbf{C}^n \to \mathfrak{k}^*$ by zero to a map on $\mathbf{C}^n \times \mathbf{C}$, restricting the extension to the sphere $\phi^{-1}(0)$ and pushing it down to a map on the quotient $\mathbf{C}P^n(r)$.

To get the identification of $N(r)$ with $\mathbf{C}P^n(r)$ we start out by embedding $\bar{B}(r)$ into $\phi^{-1}(0)$ via the map

$$f : z \mapsto \left(z, \sqrt{r - |z|^2}\right).$$

Composing f with the orbit map $\phi^{-1} \to \phi^{-1}/U(1)$ we get a map f' from $\bar{B}(r)$ *onto* $\mathbf{C}P^n(r)$. It is easy to see that f' descends to a homeomorphism f'' from $N(r)$ to $\mathbf{C}P^n(r)$. It is also easy to see that

$$f''(q(J^{-1}(\mathcal{O}) \cap \bar{B}(r))) = f'(q(J^{-1}(\mathcal{O}) \cap \bar{B}(r))) = J_r^{-1}(\mathcal{O}).$$

Obviously, the moment map $J_r : \mathbf{C}P^n(r) \to \mathfrak{k}^*$ is proper. So Remark 2.13 implies that the set $J_r^{-1}(\mathcal{O})$ is connected and we are done. □

5 Reduced Space at Angular Momentum Zero for n Particles in d-space

Let V be the phase space for n particles in d-dimensional Euclidean space;

$$\begin{aligned} V &= T^*\mathbf{R}^d \times T^*\mathbf{R}^d \times \dots T^*\mathbf{R}^d \quad (n \text{ times}) \\ &= \mathbf{R}^d \times \mathbf{R}^d \times \dots \mathbf{R}^d \quad (2n \text{ times}). \end{aligned}$$

Take $G = O(d)$ to be the orthogonal group associated to $\mathbf{R}^d$, with $g \in G$ acting on V according to

$$g \cdot (q_1, p^1, q_2, p^2, \dots, q_n, p^n) = (gq_1, gp^1, gq_2, gp^2, \dots, gq_n, gp^n).$$

We will use Greek indices, μ, ν, etc. for the particle labels, and Latin indices i, j etc. to index the coordinates on the Euclidean space $\mathbf{R}^d$. So V has coordinates (q^i_μ, p^ν_j), for $\mu, \nu = 1, 2, \dots, n$ and $i, j = 1, 2, \dots, d$, which shows that

$$M \cong \mathbf{R}^d \otimes \mathbf{R}^{2n}. \tag{12}$$

Under this isomorphism the G-action becomes $g(x \otimes z) = gx \otimes z$. The symplectic form on V is $\Omega = \Sigma_{i,\mu} dq^i_\mu \wedge dp^\mu_i$. The momentum map for the $O(d)$-action is

$$J(q, p) = \Sigma_\mu q_\mu \wedge p^\mu,$$

where we have used the inner product on $\mathbf{R}^d$ to identify $\Lambda^2\mathbf{R}^d$ with the Lie algebra of $O(d)$ and its dual space. Equation (12) expresses V as the tensor product of the inner product space $\mathbf{R}^d$ with the symplectic vector space $\mathbf{R}^{2n}$. Since $h \in H := Sp(n, \mathbf{R})$ acts by $h(x \otimes z) = x \otimes hz$, it is clear that the actions of $G = O(d)$ and of H commute. The momentum map for the $Sp(n, \mathbf{R})$-action is given by

$$\sigma(q, p) = \begin{pmatrix} q_\mu \cdot q_\nu & q_\mu \cdot p^\nu \\ p^\mu \cdot q_\nu & p^\mu \cdot p^\nu \end{pmatrix}. \tag{13}$$

Here '$\cdot$' denotes the inner product on $\mathbf{R}^d$: $q_\mu \cdot q_\nu = \Sigma_i q^i_\mu q^i_\nu$. Thus $S = \sigma(q, p)$ is a symmetric $2n \times 2n$-matrix which we have written in terms of four $n \times n$-blocks.

In saying that σ is the momentum map we are identifying the space $S^2(\mathbf{R}^{2n})$ of symmetric $2n \times 2n$-matrices on $\mathbf{R}^{2n}$ with the dual of the Lie algebra of $\mathfrak{sp}(n, \mathbf{R})$ since the target of the map σ is $S^2(\mathbf{R}^{2n})$. What is the identification $S^2(\mathbf{R}^{2n}) \cong \mathfrak{sp}(n, \mathbf{R})^*$? The trace pairing (Killing form) $(S_1, S_2) \mapsto \operatorname{trace} S_1 S_2$ induces an isomorphism $\mathfrak{sp}(n, \mathbf{R}) \cong \mathfrak{sp}(n, \mathbf{R})^*$. The identification of $S^2(\mathbf{R}^{2n})$ with $\mathfrak{sp}(n, \mathbf{R})$ is described by mapping S to JS where J is the symplectic operator: $J^2 = -1$, $JJ^t = I$, $\Omega(v, w) = \langle v, Jw \rangle$. Composing these identifications yields the desired one: $\mathfrak{sp}(n, \mathbf{R})^* \cong S^2(\mathbf{R}^{2n})$. Under this isomorphism the coadjoint action of $Sp(n, \mathbf{R})$ intertwines with the action $S \mapsto gSg^t$ of $Sp(n, \mathbf{R})$ on $S^2(\mathbf{R}^{2n})$.

The 'first main theorem of invariant theory' (see e.g. [25, Theorem 2.9A]) states that the entries of $S = \sigma(q, p)$ in the formula for σ form a basis for the $O(d)$-invariant polynomials on V. Consequently assumption Q of the previous section holds and so the restriction of $\bar{\sigma}$ to $J^{-1}(0)/O(d)$ is an isomorphism onto its image. (As in the previous section, $\bar{\sigma} : M/O(d) \to S^2(\mathbf{R}^2 n)$ is the map induced by σ.) What is its image?

Let $\Sigma \subset S^2(\mathbf{R}^2 n)$ denote the set of *nonnegative* symmetric matrices whose kernel is coisotropic. (This means that the kernel contains its Ω-orthogonal complement.) Let $\Sigma_k \subset \Sigma$ denote the subset of Σ consisting of matrices of rank k, and let

$$\Sigma^k = \cup_{i \leq k} \Sigma_k \tag{14}$$

denote the subset of matrices with rank at most k. As a subset of $\mathfrak{sp}(n, \mathbf{R})$ the set Σ_j is a single coadjoint orbit, and $\Sigma^k = \overline{\Sigma_k}$ is the union of $k + 1$ nilpotent orbits, these being the Σ_j, $j \leq k$, with $\Sigma_0 = \{0\}$. These are the strata of Σ^k. We will show that $\sigma(J^{-1}(0)) = \Sigma^k$ where $k = \min(d,n)$. Once this is shown we will have proven:

5.1. THEOREM. *Let V_0 denote the reduced space at angular momentum 0 for the action of $O(d)$ on the phase space V of of n particles in d-space. Then V_0 is isomorphic as a stratified symplectic space to the set Σ^k described in (14), where $k = \min(d,n)$. The isomorphism is the one induced by the $Sp(n,\mathbf{R})$-momentum map, namely the restriction of $\bar{\sigma}$ to $J^{-1}(0)/O(d)$.*

PROOF. We proved in the previous section that σ induces an isomorphism with all the desired properties. It remains to prove that the image of σ restricted to $J^{-1}(0)$ is Σ^k. We have

$$\sigma(q,p) = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \\ p^1 \\ p^2 \\ \vdots \\ p^n \end{pmatrix} \left(q_1{}^t \, q_2{}^t \dots q_n{}^t \, p^{1t} \, p^{2t} \dots p^{nt} \right).$$

From this expression, it is clear that $\sigma(q,p)$ is nonnegative. Since each of the two factors of $\sigma(q,p)$ is a matrix with rank less than or equal to d, the matrix $\sigma(q,p)$ has rank less than or equal to d.

5.2. REMARK. These are the only constraints on the image of σ. This is the content of the the 'second main theorem of invariant theory' for the orthogonal group [25, Theorem 2.17A]. However, we do not need this result to prove our theorem, as it will in our case follow from equivariance.

Now let us restrict σ to $J^{-1}(0)$. There we have $q_1 \wedge p^1 + q_2 \wedge p^2 + \dots q_n \wedge p^n = 0$. Let us assume for simplicity that the q^i are linearly independent. Then Cartan's lemma (see e.g. [22, p. 19]) states that we have $p^\mu = \Sigma S^{\mu\nu} q_\nu$ for some symmetric $n \times n$ matrix S. A direct calculation now shows that in this case

$$\sigma(q,p) = \begin{pmatrix} M & MS \\ SM & SMS \end{pmatrix}, \tag{15}$$

where $M_{\mu,\nu} = q_\mu \cdot q_\nu$ is the matrix of inner products. Note that

$$\sigma(q,p)\begin{pmatrix} S \\ -I \end{pmatrix} = 0,$$

from which it follows that the kernel of the map $\sigma(q,p)$ contains the Lagrangian subspace $\{(Sy,-y) : y \in \mathbf{R}^n\}$. But any subspace containing a Lagrangian one is coisotropic, so we have proved our result in this particular case.

In general the q^i are not linearly independent. But a slight variant of the proof of Cartan's lemma shows us that the dimension of the space spanned by $\{q_1, q_2, \dots, p^1, p^2, \dots, p^n\}$ is less than or equal to n. It follows from the factorization of σ that the rank of $\sigma(q,p)$ is always less than or equal to n. We have proved the statement regarding the rank of the matrices in $\sigma(J^{-1}(0))$.

A few moments' reflection should convince the reader that each Σ_j is a single orbit of the $Sp(n,\mathbf{R})$-action on $S^2(\mathbf{R}^2 n)$. Hint: Write $\mathbf{R}^{2n} = L_1 \oplus L_2$ where the L_i are Lagrangian subspaces and $A \in \Sigma_j$ annihilates L_2. Note that $Sp(n,\mathbf{R})$ acts transitively on pairs (L_1, L_2) of transverse Lagrangian subspaces, and that, relative to this splitting $g \oplus g^t \in Sp(n,\mathbf{R})$ for any $g \in Gl(L_1)$. (The symplectic form allows us to identify L_2 with the dual of L_1.) Now suppose that we can show that there is *some* matrix $A \in \Sigma_k \cap \sigma(J^{-1}(0))$. Then it follows from the Sp-equivariance of σ and the Sp-invariance of J that $\Sigma_k \subset \sigma(J^{-1}(0))$. It is also clear that the closure of Σ_k is Σ^k. The map σ, being homogeneous and quadratic, is a closed map. It now follows from $\Sigma_k \subset \sigma(J^{-1}(0))$ that $\sigma(J^{-1}(0) = \Sigma^k$ as desired. Thus all we have to do is produce a single matrix A in Σ_k which we can be written in the form $\sigma(q,p)$ for some $(q,p) \in J^{-1}(0)$. Take $(q,p) = (q,0)$. If $d \geq n$, set $q = (e_1, e_2, \dots e_n)$, the first n elements of an orthonormal basis $\{(e_1, e_2, \dots, e_d\}$ for $\mathbf{R}^d$. Then (see (15))

$$\sigma(q,p) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where I is the $n \times n$ identity matrix. This proves the theorem for the case $d \geq n$. In case $d < n$, take $q = (e_1, e_2, \dots, e_d, 0, \dots, 0)$. Then $\sigma(q,p)$ again has the above form, except now I is the $d \times d$-identity matrix. □

5.3. REMARK. The dual pair just discussed, $(O(d), Sp(n,\mathbf{R}))$, is the subject of [2]. See also [9] and [10, pp. 501–507].

5.4. REMARK ($O(d)$ versus $SO(d)$). Suppose, in the above discussion, that we replace $O(d)$ by the special orthogonal group $SO(d)$. Then the corresponding reduced space will be a *branched double cover* over the $O(d)$-reduced space. This is because $O(d)/SO(d)$ is the two-element group. Assumption Q fails for the group $SO(d)$. Thus we cannot use dual pairs alone to construct its reduced space. The additional, nonquadratic invariants are the d-ple products $\det[v_1, \dots, v_d]$, where the v_i are any of the vectors $q_1, \dots, p^n$. They satisfy the relation $\det[v_1, \dots, v_d]^2 = \det[v_i \cdot v_j]$. In the special case $d = 2$, the d-ple product is quadratic and we can realize the reduced space via dual pairs. Let us consider the case of our example in Section 1: $d = 2$, $n = 1$. The invariants were written down in Section 1.3 as $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. σ_3 is the 2-ple product, i.e., the signed area. The other invariants are $O(2)$-invariants. There is one relation, equation (3). It is quadratic in σ_3, explicitly showing how the $SO(2)$-reduced space is a branched double cover of the $O(2)$-reduced space.

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