Engel deformations and contact structures

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1 Introduction and Outline

An engel structure is a maximally non-integrable two-plane field peculiar to dimension four. Engel structures are topologically stable in the same way that contact and symplectic structures are stable. Namely, any two such structures are locally diffeomorphic, and the typical perturbation of an arbitrary germ of a two-plane field on a four-manifold (read "one-form on an odd-dimensional manifold" or "two-form on an even-dimensional manifold") will be engel (respectively "contact" or "symplectic"). The fact that any two engel structures are locally diffeomorphic is contained in the the "Darboux theorem" for engel structures, attributed to Engel by Cartan [5]. This normal form theorem asserts that any engel structure is locally diffeomorphic to $\mathbb{R}^4$ with coordinates x, y, z, w and with its two-plane field defined by the vanishing conditions

$$dz - ydz = 0,$$

$$dy - wdx = 0.$$

If a k-plane field in n-dimensions is stable in the above sense, then $k(n-k) \leq n$. (See [19] or [16].) The only positive integer solutions to this inequality are $k = 1$, n arbitrary, $k = n-1$, n arbitrary, or $k = 2, n = 4$. The first possibility, $k = 1$, corresponds to line fields. The second possibility $k = n-1$ is realized by contact structures when n is odd, and when n is even by an even-dimensional cousin of contact structures which we will call EVEN-CONTACT STRUCTURES, following Ginzburg [9]. These are discussed in some detail in section 2.3 below. The final possibility $(k, n) = (2, 4)$ is realized by the case of interest to us, the engel structures. The uniqueness of engel structures within this list, and the special appearance of dimension four are primary reasons for studying them.

In this paper we will study deformations of engel structures. Our investigations have been guided by analogy with contact structures. A basic theorem for contact manifolds is the theorem of Gray [11]. This asserts that if two contact structures ξ_0, ξ_1 are homotopic through intermediate contact structures ξ_t then the two structures are in fact diffeomorphic via an isotopy. We will see in theorem 4 of the present paper that the analogous theorem is false for engel structures.

A major difference between engel structures and contact structures is the existence of a CANONICAL FLAG

$$L \subset \mathcal{D} \subset \mathcal{E} \subset TQ$$

of distributions attached to any engel distribution. Here Q is the manifold, TQ its tangent bundle, and $\mathcal{D} \subset TQ$ is the engel structure. $\mathcal{E}$ is the rank 3 subbundle of TQ which is formed (locally) by taking the span of the Lie bracket of any two linearly independent vector fields tangent to $\mathcal{D}$ and adding this to $\mathcal{D}$. L is a canonical line subbundle of $\mathcal{D}$ constructed as the kernel of some map based on the Lie bracket and $\mathcal{E}$. See section 2.3 for details.

When we perturb $\mathcal{D}$, we will perturb its line field. But line fields are typically **not** structurally stable: small perturbations lead to non-diffeomorphic fields. Hence we expect that the analogue of Gray's theorem will fail in the engel context. This accounts for theorem 4. The idea just outlined can be found in Gershkovich [8], but the proof there appears incomplete and we have completed it. If one insists that the line field remain fixed during the engel homotopy, then the analogue of Gray's theorem can be salvaged. Golubev [10] did this after the initial write-up of the present paper.

The main topic of the present paper is deformations of certain canonical engel structures on four-manifolds of the form $M^3 \times S^1$ or $M^3 \times I$ where M^3 is a contact three-manifold. We will prove in theorems 4 and 5 that the space of deformations is infinite dimensional. This is to be contrasted with the case of contact manifolds, where the deformation is discrete, according to Gray's theorem.

The central tool for obtaining this result is an an interplay between contact three-manifolds and engel structures which is summarized by theorems 1-3. These theorems are well-known to a few experts, notably R. Bryant and others familiar with the works of E. Cartan. In one form or another they are all contained in the works of E. Cartan. They are central to our work. We could not find clear, modern statements of them, so we reformulate and prove them here. We do not make any claim to originality in theorems 1-3 .

Theorem 1 constructs an engel structure out of any contact three-manifold. Cartan called this construction 'prolongation' and applied it to general Pfaffian systems. Theorem 2 concerns the inverse of prolongation. This process allows us to associate a "local contact manifold" (more accurately, a transverse contact struture) to an engel structure. This local manifold is the "local quotient" of the four-manifold by the integral curves of the engel line field L. Through this description one also finds that the leaves of L inherit a real projective structure. Bryant and Hsu explicity described this real projective structure in their article [4]. Theorem 3 concerns the relationship between engel and contact automorphisms. Theorems 1-3 may be summarized as saying that engel geometry is a twisted product of three-dimensional contact and one-dimensional real projective geometries.

We end the paper with theorem 6 where we construct an engel manifold whose automorphism group is "very small"- essentially one-dimensional. This is to be contrasted with the contact case where the automorphism group is always infinite-dimensional. This theorem is proved by combining theorems 3 and 4.

When I began these investigations I had hoped that they might serve as a first step towards the development of an "engel topology". Theorem 4 and 6 and the ideas around them have killed much of this hope. The engel line field dominates global investigations. It appears that any deep global results on engel structures will involve a strange mix of topology and dynamical systems.

2 Preliminaries.

2.1 Definitions

Let Q be an n-dimensional manifold and TQ its tangent bundle. By a k-plane field or DISTRIBUTION on Q we mean a smooth linear subbundle $\mathcal{D} \subset TQ$. The integer k is the rank of the subbundle. We can think of $\mathcal{D}$ as a locally free sheaf of smooth vector fields on the manifold. Then $[\mathcal{D}, \mathcal{D}]$ will denote the sheaf generated by all Lie brackets $[X, Y]$ of sections X, Y of $\mathcal{D}$. Set $\mathcal{D}^2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ and continue to take Lie brackets, setting $\mathcal{D}^{j+1} = \mathcal{D}^j + [\mathcal{D}, \mathcal{D}^j]$. (One can check that $\mathcal{D}^j \subset [\mathcal{D}, \mathcal{D}^j]$ as sheaves, so that it is in fact unneccessary to add $\mathcal{D}^j$ in defining $\mathcal{D}^{j+1}$.) Write $\mathcal{D}^j(x) \subset T_xQ$ for the subspace obtained by evaluating all the vector fields in $\mathcal{D}^j$ at the point x. Write $n_j(x) = dim(\mathcal{D}^j(x))$. We will also write $n_j(x; \mathcal{D})$ when we want to indicate the dependence of the integers on the distribution.

Definition 1 *$\mathcal{D}$ is called* NONHOLONOMIC *if for all $x \in Q$ there is a $j = j(x)$ such that $\mathcal{D}^j(x) = T_xQ$.*

$\mathcal{D}$ is called MAXIMALLY NON-HOLONOMIC *if for each j we have $n_j(x; \mathcal{D}) = max_{\mathcal{E}} n_j(x; \mathcal{E})$ where the maximum is taken over all rank k distributions $\mathcal{E}$ defined in some neighborhood of x.*

Gershkovich and Vershik [19] call the list of integers $n_j(x)$ the GROWTH VECTOR of $\mathcal{D}$ at x. The list of integers $max_{\mathcal{E}} n_j(x; \mathcal{E})$ is then called the MAXIMAL GROWTH VECTOR for the given k and n. The components of the maximal growth vector are of course independent of x. They equal the dimensions f_j of the graded components of the free Lie algebra on k generators, up until $f_j \geq n$. In case $(k, n) = (2, 4)$ the maximal growth vector is $(2, 3, 4)$.

Definition 2 *An* ENGEL FIELD, *or* ENGEL STRUCTURE *on a four-manifold Q is a rank two subbundle $\mathcal{D} \subset TQ$ whose growth vector is everywhere maximal, and consequently equals $(2,3,4)$. An* ENGEL MANIFOLD *is a four-manifold with an engel structure.*

Thus if $\mathcal{D}$ is an engel structure then $\mathcal{D}^2$ is a rank 3 distribution and $\mathcal{D}^3$ is the entire tangent bundle. More concretely, an engel structure on a four-manifold is a two-plane field which is locally spanned by vector fields $\{X, W\}$, such that $\{X, W, [X, W], [X, [X, W]]\}$ frame the whole tangent bundle in the neighborhood in which two fields are defined.

Any two engel structures are locally diffeomorphic, as mentioned in the beginning.

2.2 Prolongations

Let $\mathcal{E}$ be a distribution of k-planes on a manifold M. Imagine that we are interested in ℓ-dimensional submanifolds, $\ell < k$, of M which are tangent to $\mathcal{E}$. Such an study suggests forming the bundle $Q = Gr_\ell(\mathcal{E}) \to M$ whose fiber over $m \in M$ consists of the Grassmannian of all ℓ-dimensional linear subspaces $P \subset \mathcal{E}_m$. These ℓ-planes P represent the possible tangent spaces to the submanifolds. A curve in $Gr_\ell(\mathcal{E})$ consists of a moving pair $(m(t), P(t))$ with $P(t) \subset \mathcal{E}_{m(t)}$. Define a distribution on the manifold $Gr_\ell(\mathcal{E})$ by declaring that a curve $(m(t), P(t))$ is tangent to the distribution if and only if $\frac{dm}{dt} \in P(t)$. Alternatively, the distribution plane at the point (m, P) is $d\pi^{-1}_{(m,P)}(P)$ where $\pi : Gr_\ell(\mathcal{E}) \to M$ is the projection and $d\pi$ its differential.

Definition 3 *The total space of this Grassman bundle, together with the distribution just constructed, is called a* PROLONGATION *of $\mathcal{E}$.*

This construction is due to Cartan [5]. See also the final section of [3], and the other references to Cartan therein.

EXAMPLE. Let us apply prolongation to the case where $\mathcal{E} = TM$ is the entire tangent bundle of the n-dimensional manifold M, and where $\ell = n - 1$, as would be the case if we were interested in studying hypersurfaces in M. The resulting prolongation of the tangent bundle is canonically isomorphic to $I\!PT^*M$, since the vanishing of a non-zero one-form $\theta_m \in T^*_m M$ defines a hyperplane in the tangent space at m. The distribution on $I\!PT^*M$ constructed by the prolongation process has rank $2n - 2$, compared with the dimension $2n - 1$ of $I\!PT^*M$. It is a contact structure. Indeed, this example explains the origin of the word "contact": a linear hyperplane within the tangent space at a point is called a "contact element" at that point, representing, as it does contact with an alleged hypersurface passing through that point.

When we prolong with $\ell = 1$, the case corresponding to curves tangent to $\mathcal{E}$, then the total space of the Grassmann bundle is the projectivization $I\!P\mathcal{E}$ of $\mathcal{E}$. The fiber $I\!P\mathcal{E}_m$, $m \in M$ is the real projective space of dimension $k-1$ consisting of the one-dimensional subspaces $\ell \subset \mathcal{E}_m$ of the vector space $\mathcal{E}_m$. Observe that if $\mathcal{E}$ has rank 2, then this process of prolonging with $\ell = 1$ increases the dimension of the space by 1, but keeps the rank of the distribution the same.

Theorem 1 *The prolongation to $I\!P\xi$ of a contact structure $\xi \subset TM$ on a three-manifold yields an engel structure on the four-manifold $I\!P\xi$.*

REMARK. Curves tangent to a contact structure on a three-manifold are called "legendrian curves". So we might call $I\!P\xi$ the space of LEGENDRIAN CONTACT ELEMENTS for ξ.

PROOF. The Darboux theorem applied to the contact structure ξ yields coordinates (x, y, z) on M centered at any preassigned point $m \in M$ with the property that $\xi = \{dz - ydx = 0\}$ near m. The one-forms dx and dy are linear coordinates on each of the contact planes $\xi_{x,y,z}$ near m. Then $[dx : dy]$ form homogeneous coordinates on the corresponding projective line $I\!P\xi_{(x,y,z)}$. Write $dy/dx = w$ or $dy = wdx$ for the corresponding affine coordinate w. Then (x, y, z, w) are coordinates for the prolongation. The Pfaffian equations

$$\theta_1 := dz - ydx = 0$$

and

$$\theta_2 := dy - wdx = 0$$

define the prolonged distribution. Indeed, the first equation asserts that the moving point $m(t)$ with coordinates (x, y, z) is tangent to ξ and the second equation asserts that this moving point is tangent to the line $\ell \subset \xi$ represented by the affine fiber coordinate w. **Note that the above representation is the normal form presented at the beginning of this article.**

The vector fields

$$W = \frac{\partial}{\partial w}$$

and

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} + w\frac{\partial}{\partial y}$$

span the prolonged distribution, as they are annihilated by θ^1 and θ^2. We compute $[W, X] = \frac{\partial}{\partial y}$ and $[X, [W, X]] = \frac{\partial}{\partial z}$. Now $W, X, [W, X]$ and $[X, [W, X]]$ span $I\!R^4$ everywhere in this coordinate neighborhood. It follows that the prolonged distribution on $I\!P(\xi)$ is indeed engel, as defined by definition 1, and as elucidated in the paragraph following that definition.

QED

As mentioned in the beginning, the ENGEL NORMAL FORM asserts that in a neighborhood of any engel manifold one can find coordinates (x, y, z, w), called ENGEL COORDINATES, with the above properties. Namely, the coordinate one-forms θ^1, θ^2 above define the distribution $\mathcal{D}$ by annihilating it. We call these one-forms the CANONICAL CO-FRAME. The vector fields W, X written above and spanning $\mathcal{D}$ will be called THE CANONICAL FRAME in these coordinates. A modern derivation of the engel normal form can be found in [2] where it is a special case of the "Goursat normal form".

Corollary 1 *If $Q^4 = M^3 \times S^1$ with M^3 oriented, then Q^4 admits an engel structure.*

PROOF OF COROLLARY. Martinet [14], proved that any oriented M^3 admits a contact structure ξ. Lutz [13] extended Martinet's result by showing that ξ could be taken to represent any possible homotopy class of two-plane fields which arise within TM^3. (See Eliashberg [7] for more discussion.) In particular, the trivial class can be represented, in which case $I\!P\xi$ is is diffeomorphic to $M^3 \times S^1$. QED

2.3 The Characteristic line field and Even-Contact Structures

An engel distribution $\mathcal{D}$ determines a flag

$$L \subset \mathcal{D} \subset \mathcal{E}$$

of subbundles of the tangent bundle. Here

$$\mathcal{E} := \mathcal{D}^2 = [\mathcal{D}, \mathcal{D}]$$

is the rank three distribution which represents the '3' occuring in the growth vector $(2, 3, 4)$ in the definition of being engel. $\mathcal{E}$ is spanned by $\mathcal{D}$ and the Lie bracket of any two vector fields which span $\mathcal{D}$. The line field L is defined by the condition

$$[L, \mathcal{E}] \subset \mathcal{E}.$$

It will be called the CHARACTERISTIC LINE FIELD, or sometimes the ENGEL LINE FIELD. Its integral curves will be called CHARACTERISTIC LEAVES.

In terms of engel coordinates $\mathcal{E}$ is spanned by the canonical vector fields X, Y, W. L is spanned by W alone. $\mathcal{E}$ is annihilated by $\theta_1 = dz - ydx = 0$. L is annihilated by θ_1, $d\theta_1$, and θ_2.

EXAMPLE. If $Q = I\!P\xi$ is the prolongation of a contact three-manifold (M, ξ) with the engel structure of theorem 1, then its characteristic leaves are the circular fibers $IP(\xi_m)$, $m \in M$, of the fibration $Q \to M$. $\mathcal{E}$ is the pull-back of ξ by π.

$\mathcal{E}$, considered by itself, is an EVEN-CONTACT STRUCTURE.

Definition 4 *An* EVEN-CONTACT *structure (sometimes called a "quasi-contact structure") is a field $E \subset TQ$ of hyperplanes on an* **even-dimensional** *manifold Q with the property that if θ is a one-form whose vanishing defines E (perhaps only locally), then the pointwise restrictions $d\theta|_{E_q}$ are two-forms of maximal rank on the E_q.*

If we had insisted instead that the manifold were odd-dimensional then we would have been giving the definition of a contact manifold.

We will now show that in our flag above, $\mathcal{E}$ determines L, without the knowledge of $\mathcal{D}$. A two-form on an odd-dimensional vector space always has a kernel. The even-contact condition is that the kernel of the $d\theta|_{E_q}$ on E_q be of minimal dimension, and consequently of dimension one. These kernels define a line field $L_q \subset E_q$, $q \in Q$, which we call the characteristic line for E. In dimension four the even-contact condition is equivalent to the condition $[E, E] = TQ$, so the $\mathcal{E} = \mathcal{D}^2$ of an engel structure is indeed even-contact. Now any rank 2 subbundle F of an even-contact structure E with the property that $[F, F] \subset E$ must necessarily contain L. **From these observations it follows that the characteristic line field L of an engel structure is determined completely by the even-contact structure $\mathcal{D}^2$.**

There is a Darboux theorem for even-contact structures. In dimension four it yields the normal form

$$\theta = dz - ydx$$

for the one-form whose vanishing defines the even-contact structure $\mathcal{E}$. (Note this is precisely the form θ_1 of the engel normal form above.) The four-manifold coordinates are x, y, z, w and the field $W = \frac{\partial}{\partial w}$ spans the characteristic line field L. This line field L is uniquely determined by the two conditions $\theta(L) = 0$, and $d\theta(L, \cdot) = 0$. Together, these equations say that the flow of any vector field tangent to L preserves E:

$$[L, E] \subset E.$$

For $U \subset Q$ an open set of an even-contact manifold, write U/L for the quotient space whose points are connected leaves of the characteristic line field L in U. Suppose that this quotient is "nice" in the sense that it is a three-manifold and that the natural projection $\pi : U \to U/L$ is a submersion. Then the quotient inherits a contact structure, denoted $\pi_* E$. Indeed, the flow along L leaves E invariant, so that we can push E down along L. In other words, the subspaces $d\pi_q E_q$ and $d\pi_p E_p$ are equal for q, p on the same leaf of L. This family of subspaces has dimension 2 since $L \subset E$ and we are dividing out by L.

The condition $[E, E] = TU$ implies that $[\pi_* E, \pi_* E] = T(U/L)$. So the quotient distribution $\pi_* E$ is contact.

If U is the neighborhood for a canonical coordinate system (x, y, z, w) then (x, y, z) coordinatize the quotient space. The contact form for $\pi_* E$ has the same expression $dz - ydx$ as the even-contact form had on U.

If $\Sigma \subset M$ is a three-manifold transverse to L, then we may take $U = N(\Sigma)$ to be a tubular neighborhood of Σ swept out by flowing along L for short times. Then the quotient U/L is canonically isomorphic to Σ. And the contact structure on the quotient is identified in this way with $E \cap T\Sigma$.

These observations prove (a) and (b) of the following theorem.

Theorem 2 *Let $\mathcal{E}$ be an even-contact structure on the four-manifold Q, and L its characteristic line field.*

a) If the leaf space Q/L is 'nice' in the above sense then it inherits a contact structure $\pi_*\mathcal{E}$ from $\mathcal{E}$.

b) If $\Sigma \subset Q$ is a three-manifold transverse to the L then $\mathcal{E}|_\Sigma := \mathcal{E} \cap T\Sigma$ is a contact structure on Σ.

Suppose now, that $\mathcal{E} = \mathcal{D}^2$ is the even-contact structure associated to an oriented Engel structure $\mathcal{D}$. Then $\mathcal{D} \cap T\Sigma \subset \mathcal{E}_\Sigma$ is a line sub-field, and hence a global section of $I\!P(\mathcal{E}|_\Sigma) \to \Sigma$. We identify the image of this section with Σ.

c) Any sufficiently small neighborhood $N(\Sigma)$ of Σ in Q is diffeomorphic, as an engel manifold, to some neighborhood of Σ in $I\!P(\mathcal{E}|_\Sigma)$ with its prolonged engel structure as described in theorem 1.

EXAMPLE. Take $Q = I\!P\xi$ to be the prolongation of a contact three-manifold (M, ξ) as in Theorem 1. Then $Q/L = M$ and the induced contact structure is the original contact structure ξ.

Before we go on to the proof of the theorem, we will rephrase its assertions (a) and (b). They assert that an even-contact structure on a four-manifold induces a TRANSVERSE CONTACT STRUCTURE on that manifold. Let us pause to define the notion of transverse structure generally. Let L be a foliation of an n-manifold by s-dimensional leaves. Let P denote some type of geometric structure, eg. contact structure, symplectic structure, volume form, metric, ..., which may exist on manifolds of the complementary dimension $n - s$.

Definition 5 *We say that the foliation L of Q has a* TRANSVERSE GEOMETRIC STRUCTURE *of type P if*

- *i) If any sufficiently small submanifold Σ transverse to the leaves of L and of complementary dimension to L is endowed with a P-structure*

- *ii) The flows along the leaves of L preserve these P-structures. More precisely, if Σ_1 and Σ_2 are two such transverse submanifolds, and if X is any vector field tangent to L whose time t flow maps Σ_1 to Σ_2 then this flow defines a P-isomorphism between $\Sigma_1 \to \Sigma_2$.*

Suppose that T is a tensor on Q whose restriction to any transverse submanifold Σ induces a structure of type P on Σ. Then T defines a transverse P-structure on Q **provided** the restrictions to the Σs of all of the Lie derivatives $L_X T$, X tangent to L, are zero. This is precisely our situation in the even-contact case. Take T to be the one-form θ whose vanishing defines the even-contact structure.

2.3.1 Twisting and the projective structure on leaves.

An engel structure is sandwiched between its even-contact structure $\mathcal{E}$ and its characteristic line field L. Almost any two-plane field $\mathcal{D}$ sandwiched between an even-contact structure and its line field will be engel. Indeed the fact that $L \subset \mathcal{D}$ implies that $[\mathcal{D}, \mathcal{D}] \subset \mathcal{E}$. To make sure $\mathcal{D}$ is engel we must impose the single nondegeneracy condition $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$. Since $\mathcal{D}$ is rank 2 this is equivalent to the TWISTING CONDITION:

$$[L, \mathcal{D}] = \mathcal{E}.$$

An engel structure is two-plane field sandwiched between an even-contact structure and its line field, and which "twists" about the line field in the above sense.

2.3.2 The development map.

Now let U be a 'nice' neighborhood for L, so that $U \to U/L$ is a submersion. Let $\xi = \pi_* \mathcal{E}$ be the induced contact structure. A point $m \in U/L$ represents an integral curve $\ell \subset U$ for L. At each point $q \in \ell$ we have the two-plane field $\mathcal{D}_q \subset \mathcal{E}_q$, with $L_q \subset \mathcal{D}_q \subset \mathcal{E}_q$. Consequently $d\pi_q(\mathcal{D}_q) \subset \pi_*(\mathcal{E})_m$ is a line, or one-dimensional subspace, of the two-plane ξ_m. The set of all such lines forms the real projective line $\mathbb{R}\mathbb{P}^1 = \mathbb{P}(\xi_m)$. In this manner we obtain a smooth map $\ell \to \mathbb{P}(\xi_m)$. We call this map, or its lift to the corresponding universal covers, the development map along the leaf ℓ. The "twisting" discussion above implies that the condition that $\mathcal{D}$ be engel is equivalent to the condition that these development maps be local diffeomorphisms for all leaves. In other words, as we climb up any given leaf ℓ, the lines $d\pi_q(\mathcal{D}_q) \cong \mathcal{D}_q / L_q$ always rotate, and in the same sense, within the circle $\mathbb{P}(\xi_m)$.

Definition 6 *The map $\ell \to \mathbb{P}(\xi_m)$ restricted to a leaf of L , or the union $U \to \mathbb{P}(\xi)$ of these maps over the leaves ℓ lying in some 'nice' neighborhood U, will be called the* DEVELOPMENT MAP *associated to the engel structure.*

2.3.3 Proof of theorem 2.

We had proved (a) and (b) just before stating the theorem. It remains to prove (c). In proving (b) we observed that the quotient $N(\Sigma)/L$ was diffeomorphic to Σ itself. The development map defines a map $N(\Sigma) \to I\!P(\xi)$, $\xi = \mathcal{E}|_\Sigma$. Indeed, if we use L to make $N(\Sigma)$ into a fiber bundle, with fibers equal to intervals, then this map is a bundle map over the projections to Σ, and its restriction to each leaf ℓ has nonzero derivative. Consequently, the development map is a diffeomorphism onto its image, provided $N(\Sigma)$ is taken small enough. The coordinate description of the quotient procedure, and the use of w as affine coordinate on the lines $I\!P(\xi_m)$ shows that this map preserves Engel structures. QED

2.4 Summary

The characteristic line field of an engel manifold admits a transverse contact structure, and the individual leaves of this line field admit a real projective structure.

3 Automorphisms

How do contact and engel automorphisms relate to each other?

Theorem 3 (Automorphism Theorem) *(a) Let $I\!P\xi$ be the engel structure obtained by prolonging the contact three-manifold (M, ξ) as in theorem 1. Then every engel automorphism of $I\!P\xi$ is induced by a contact automorphism of M.*

In (b) and (c) below, we suppose that Q is an engel manifold which admits a GLOBAL SLICE *Σ, by which we mean a three-manifold which intersects every leaf of of the engel line field L transversally.*

(b) Any engel automorphism $F : Q \to Q$ is uniquely determined by its restriction to Σ, this being a contact map from Σ to its image $F(\Sigma)$.

(c) If the engel automorphism F is close to the identity then by using the flow along L we can define a "Poincare return map" $\phi_F : \Sigma \to \Sigma$ associated to F. This return map is a contact automorphism of Σ which uniquely determines F.

PROOF.

(a) Any contact diffeomorphism $\phi : M \to M$ induces an engel diffeomorphism ϕ_* of $I\!P\xi$ by differentiation: $\phi_*(m, \ell) = (\phi(m), d\phi_m(\ell))$.

Now suppose that F is an arbitrary engel automorphism. Since the construction of L and $\mathcal{E} = \mathcal{D}^2$ depend only on $\mathcal{D}$, the map F must also preserve L and $\mathcal{E}$. Consequently it induces a contact map ϕ on the leaf space $M = I\!P(\xi)/L$, and this in turn induces the engel diffeomorphism $\Phi = \phi_*$ We claim $F = \Phi$.

$G = F^{-1} \circ \Phi$ is an engel automorphism which induces the identity automorphism on M. It suffices to show all such automorphisms are the identity on $I\!P(\xi)$. In engel coordinates such a map G has the form $(x, y, z, w) \mapsto (x, y, z, g(x, y, z, w))$. For it to be an engel automorphism we must have $G^*\theta_1 = 0 \bmod \{\theta_1, \theta_2\}$, and $G^*\theta_2 = 0 \bmod \{\theta_1, \theta_2\}$, where $\theta_1 = dz - ydx, \theta_2 = dy - wdx$ are the canonical forms whose vanishing defines $\mathcal{D}$. Thus

$$dy - gdx = a\theta_1 + b\theta_2$$

for some functions a, b. Since no dz's occur on the left hand side we have $a = 0$. Now write $g = w + h$. We obtain the equation $\theta_2 - hdx = b\theta_2$. Considering this equation mod θ_2, we obtain $h = 0$ so that $g = w$ and G is the identity.

(b). Let $F, G : Q \to Q$ be two engel automorphisms. The claim is that if $F(x) = G(x)$ for all $x \in \Sigma$ then $F = G$. Again, consider $F \circ G^{-1}$. This is an automorphism which is the identity on Σ. According to theorem 2 we can identify a neighborhood of Σ with a neighborhood of $I\!P\xi$ where $\xi \subset T\Sigma$ is the induced contact structure on Σ. Now apply result (a) to conclude that $F \circ G^{-1}$ is the identity in some neighborhood of Σ.

To show that $F = G$ everywhere we invoke the fundamental theorem of projective geometry. A projective transformation of the real projective line which fixes three points is the identity. In particular if such a transformation is the identity on some interval of the line then it is the identity. This last assertion is valid for any connected one-dimensional manifold with a real projective structure, and hence for the leaves ℓ of L. Being engel, $F \circ G^{-1}$ preserves the real projective structure on each leaf ℓ. But we have seen that $F \circ G^{-1}$ is the identity on an interval of every leaf. Thus $F = G$ everywhere.

(c): We represent a tubular neighborhood of Σ by a neighborhood of the zero section in the normal bundle $N(\Sigma)$, in such a way that the fibers of the normal bundle correspond to the leaves ℓ of L. If F is sufficiently C^1- close to the identity, then $F(\Sigma)$ can be represented as the image of a section $s : \Sigma \to N(\Sigma)$. We can then "push" this image down along the fibers of $N(\Sigma)$, homotoping the image back to the zero section. This homotopy corresponds to flowing along the leaves of L, and hence to the restriction of an **even-contact** automorphism $\Phi : Q \to Q$ to $F(\Sigma)$. **Warning: this flow Φ is not an engel automorphism.** Our 'Poincare return map' is $\phi_F := \Phi^{-1} \circ F|_\Sigma$, a contact automorphism of Σ. According to (c) of theorem 2, our tubular neighborhood is engel-diffeomorphic to a neighborhood in $I\!P(\xi)$. According to (a) just proved, ϕ_F determines F in this neighborhood, and according to (b) just proved this determines F everywhere. QED

3.1 Coordinate form of automorphisms.

Let (x, y, z, w) be canonical engel coordinates and X, Y, Z, W be the canonical frame as before. Write $X[f] = df(X)$ for the directional derivative of f in the direction of the vector field X. Then any infinitesimal engel automorphism V has the form:

$$V = Y[f]X - X[f]Y + fZ + X^2[f]W$$

for some smooth function $f = f(x, y, z)$ **independent of** w. This formula can be derived by expanding out V in terms of the frame, and computing the Lie derivative L_V of the defining one-forms θ_1 and θ_2. Note that the projection of the vector field V to the space of x, y, z variables is well-defined, and is an infinitesimal contact automorphism, as it should be.

4 Deformations

We are ready to move on to our main study, that of deformations of standard engel structures.

Definition 7 *A* DEFORMATION *of an engel structure $\mathcal{D}$ is a one-parameter family of engel structures $\mathcal{D}_t$, $t \in I \subset I\!R$, which passes through $\mathcal{D}$ at $t = 0$: $\mathcal{D}_0 = \mathcal{D}$. The $\mathcal{D}_t$ vary smoothly with t. The interval I of deformation parameters may be any interval containing 0 in its interior.*

Fix a contact three-manifold (M, ξ) where ξ is a globally trivial contact structure. Split

$$\xi = V_0 \oplus V_1$$

into the sum of two legendrian line fields, V_0, V_1. Then $I\!P\xi \cong M \times S^1$ where (m, θ) corresponds to the legendrian line spanned by the vector

$$v(\theta, m) = cos(\theta)v_0(m) + sin(\theta)v_1(m)$$

where the v_i are non-vanishing vector fields spanning V_i. This circle S^1 has circumference π, since v and $-v$ represent the same line.

Consider the domain $\Omega = \Omega(V_0, V_1)$ corresponding to the constraint $0 \leq \theta \leq \pi/2$. This consists of all legendrian lines lying "between" V_0 and V_1.

Definition 8 *We call $\Omega(V_0, V_1) \subset I\!P(\xi)$, with its induced Engel structure, the* STANDARD DOMAIN *associated to the pair of legendrian line fields (V_0, V_1).*

$\Omega(V_0, V_1)$ is a four-dimensional submanifold with boundary. Its boundary has two components, one corresponding to V_0 and the other to V_1. Being sections, each boundary component is an embeddded copy of M. $\Omega(V_0, V_1)$ is diffeomorphic to $M \times I$, $I = [0, \pi/2]$.

Theorem 4 *(a) Let $\mathcal{D}_t$ be any sufficiently small deformation of the canonical engel structure on the standard domain $\Omega(V_0, V_1)$. Then there is a one-parameter family of engel diffeomorphisms $\phi_t : (\Omega(V_0, V_1), \mathcal{D}_t) \rightarrow$*

$(\Omega(V_{0,t}, V_{1,t}), \mathcal{D}_0)$. *In other words, we can straighten out the deformed engel structure at the expense of deforming the boundaries,* $V_{0,t}, V_{1,t}$ *of the domain.*

(b) Let $\mathcal{D}_t$ *be a small deformation of the engel structure on the whole of* $I\!P\xi \cong M \times S^1$, *where* ξ *is a parallelizable contact structure on* M^3. *Let* L_t *be its deformed line field, and* ξ_t *the deformed contact structure on* M, *induced by viewing* M *as a slice in* $I\!P\xi$. *Then the Poincare return map for* L_t *is a contact map* $\phi_t : (M, \xi_t) \to (M, \xi_t)$.

(c) Any near-identity contact isotopy of (M, ξ) *to itself can be realized, as in (b), by some engel deformation* $\mathcal{D}_t$.

This theorem, combined with the previous one, allows us to explore the space of moduli of engel deformations of $M \times I$.

Definition 9 *Two engel deformations* $\mathcal{D}_t$ *and* $\mathcal{E}_t$ *of an engel structure* $\mathcal{D}$ *on* Q *represent the same* DEFORMATION GERM *of* $\mathcal{D}$ *if there is an isotopy* $\psi_t : Q \to Q$ *such that* $\psi_t^* \mathcal{E}_t = \mathcal{D}_t$.

In general, when we say "deformation germ" we will mean the germ in t of some deformation or homotopy h_t of a global geometric structure. Deformation germs are local in t but global in $q \in Q$, and involve dividing out by the appropriate automorphisms.

Theorem 5 *(a) The space of deformation germs of the standard domain* $(\Omega(V_0, V_1), \mathcal{D})$ *with its standard engel structure is canonically isomorphic to the space of deformation germs* $(V_{0,t}, V_{1,t})$ *of* (V_0, V_1) *of pairs of legendrian line fields on* (Σ, ξ) **modulo contact isotopies**. **This space is infinite-dimensional.**

(b) The space of deformation germs of the standard engel structure on $I\!P\xi$ *is equal to the space of deformation germs of the identity* **through contact isotopies** ϕ_t *of* (Σ, ξ) **modulo** t*-dependent conjugation:* $\phi_t \sim g_t \circ \phi_t \circ g_t^{-1}$.

REMARKS 1. Gershkovich [8] realized that the engel deformation space for $M \times S^1$ was nontrivial, and very large. He based his observations on perturbations of the line field, as we have. His proof was incomplete.

2. Since the first drafts of the current paper were written, Golubev [10] proved his theorem, mentioned in the introduction. It implies, roughly speaking, that the deformation germ of an engel field is completely determined by the corresponding deformation germ L_t of its characteristic line field.

4.1 Proof of theorem 4.

a) As $\mathcal{D}$ deforms through $\mathcal{D}_t$, its line field L deforms through L_t, and its even-contact field $\mathcal{E}$ deforms through $\mathcal{E}_t$. Identify M with its image in $I\!P(\xi)$ under the section $V_0 : M \to I\!P\xi$. This copy of M is a global slice for L, since L is the vertical bundle for the fibration $I\!P(\xi) \to M$. Consequently this M is a global slice for the nearby fields L_t. M inherits the deformed contact structures

$\xi_t = \mathcal{E}_t \cap TM$. (Theorem 2.) Now Gray's theorem [11] yields a diffeomorphism $f_t : M \to M$ taking ξ_t to $\xi = \xi_0$. Thus $f_t^* \xi_t = \xi_0$. It follows that the differential of f_t induces an engel isomorphism $F_t : I\!P(\xi_t) \to I\!P(\xi)$, where $I\!P(\xi_t)$ is given the structure defined by prolongation of ξ_t, as in theorem 1.

It follows from the proof of (c) of theorem 2 that we have an engel embedding $G_t : (\Omega(V_0, V_1), \mathcal{D}_t) \to I\!P(\xi_t)$. This is because M is a global slice for L_t, and because the leaf structure of L, and hence of nearby L_t, on $\Omega(V_0, V_1)$ is so simple – namely this foliation is diffeomorphic to the foliation by intervals $\{m\} \times I$ in $M \times I$. Composing these two engel embeddings we get an engel embedding $(\Omega(V_0, V_1), \mathcal{D}_t) \to (I\!P(\xi), \mathcal{D}_0)$. This embedding is close to the identity, hence its image must be close to $\Omega(V_0, V_1)$, with boundary components transverse to the fibers of L_0. Consequently this image is of the form $\Omega(V_{0,t}, V_{1,t})$ for some pair $(V_{0,t}, V_{1,t})$ of legendrian line fields which are perturbations of the pair (V_0, V_1).

(b) The deformation $\mathcal{D}_t$ is defined on all of $I\!P\xi$. Use the same global section $M = V_0(M)$ for the L_t. Define Poincare maps $\phi_t : M \to M$ by starting at $m \in M$ and travelling along the L_t-line passing through m until its next intersection $\phi_t(m)$ with M. This requires L to be oriented, hence the assumption on parallelizability of ξ. The map ϕ_t is a contact automorphism for the contact structure $\xi_t = \mathcal{E}_t \cap TM$, according to the discussion in section 2.3 around theorem 2.

(c) We prove that we can obtain any near-identity contact automorphism ϕ of the fixed contact structure (M, ξ) as the "Poincare section" associated to an engel deformation of the canonical engel structure. We may suppose that the contact automorphism is the time-dependent flow of some time-dependent vector field $X = X(m;\theta)$, where we also think of θ as the time variable. Specifically, ϕ can be taken to be the time $\theta = \pi$ flow for this X. (Again, the circles of $M \times S^1$ have circumference π.)

As before, write elements of $I\!P(\xi)$ as the span of vectors $cos(\theta)v_0(m) + sin(\theta)v_1(m)$, where v_0, v_1 are a pair of vector fields framing ξ. This induces a global trivialization $I\!P(\xi) \cong M \times S^1$, and under this diffeomorphism the canonical engel distribution is spanned by the vector fields $\frac{\partial}{\partial\theta}$ and $v(m,\theta) = cos(\theta)v_0(m) + sin(\theta)v_1(m)$ on $M \times S^1$. Of course, $\frac{\partial}{\partial\theta}$ spans its characteristic line field $L = L_0$ for $\mathcal{D} = \mathcal{D}_0$.

Define the perturbed engel structure by

$$\tilde{\mathcal{D}} = span\{v, W\}$$

with v as before but with

$$W = \frac{\partial}{\partial\theta} + X(m,\theta).$$

We will show that this $\tilde{\mathcal{D}}$ is indeed an engel field and that its characteristic line field is spanned by W. To begin, note that v and

$$u := -sin(\theta)v_0(m) + cos(\theta)v_1(m)$$

span ξ everywhere. Since X is an infinitesimal contact symmetry for ξ, and since v and u span ξ, we have $[X, v] = fv + gu$ for some θ-dependent functions f, g. On the other hand, $[\frac{\partial}{\partial\theta}, v] = u$, when v and u are thought of as vector fields on $IP(\xi) = M \times S^1$. Thus $[W, v] = fv + (1+g)u$. Consequently $[\tilde{\mathcal{D}}, \tilde{\mathcal{D}}]$ is rank 3 wherever $g \neq -1$. This can be arranged to happen everywhere provided X is sufficiently small. In this case $\tilde{\mathcal{D}}^2 = [\tilde{\mathcal{D}}, \tilde{D}]$ is spanned by W, u and v. Since ξ is contact, $[u, v]$ is a fourth vector field, linearly independent from W, u, v everywhere. This shows that $[\tilde{\mathcal{D}}^2, \tilde{\mathcal{D}}^2] = T(M \times S^1)$ and consequently $\tilde{\mathcal{D}}$ is indeed engel. By the same computation as used in finding $[W, v]$ above, we see that $[W, u]$ is also a linear combination of v and u everywhere. Consequently $[W, \tilde{\mathcal{D}}^2] \subset \tilde{\mathcal{D}}^2$ so that W is indeed the characteristic line field for $\tilde{\mathcal{D}}$.

By construction, the time-s flow of W is of the form $(m, 0) \mapsto (\phi_s(m), s)$, when applied to a point $(m, 0)$. Here ϕ_s is the time-s flow of the time-dependent contact vector field $X(\cdot, s)$. It follows that the return map for $\tilde{\mathcal{D}}$ is the desired contact map ϕ.

QED

4.2 Proof of theorem 5.

a) Except for the sentence about the deformation space being infinite-dimensional, the proof is a straightforward application of theorems 4 and 5.

Regarding the infinite-dimensionality we refer the reader to Cartan [6], or to Arnold's description [1] of Cartan. Cartan constructs a bijection between the germs of pairs of legendrian line fields in three-space modulo contact automorphisms and the germs of 2nd order differential equations of the form $\frac{d^2y}{dx^2} = f(y, \frac{dy}{dx}, x)$ modulo germs of diffeomorphisms of the xy plane. The latter space is known to be infinite-dimensional, and some functional moduli are described in these references.

b) The return map ϕ_t for ξ_t of the proof of part (b) of theorem 4, together with the Gray isotopy $g_t : (M, \xi_t) \to (M, \xi_0)$ yields the contact isotopy $g_t \circ \phi_t \circ g_t^{-1}$. We leave it for the reader to check that this conjugation corresponds to the correct equivalence relation in the definition of 'deformation germ'.

5 An engel structure with few symmetries.

We end with the following example.

Theorem 6 *There is a three-manifold M, and an engel structure on $M \times S^1$ with the property that the automorphism group for this structure contains exactly one one-parameter subgroup.*

Thus if we knew that the automorphism group were a Lie group then it would be one-dimensional. Contrast this state of affairs with that of global symplectic and contact structures, whose symmetries can always be localized, and consequently are always infinite-dimensional. The asymmetry in this example is due to the characteristic line field.

5.1 The construction.

The idea is to find a contact transformation $\phi = \phi_t$ for some parallelizable ξ which has very few symmetries, and then construct the corresponding engel structure following (c) of theorem 4. Our particular ϕ will be part of a one-parameter subgroup, accounting for the one-parameter group of engel automorphisms.

We begin by assuming the existence of a near-identity contact automorphism ϕ of (M, ξ), ξ parallelizable, with the following property. Firstly, ϕ is the time ϵ flow of some contact vector field X. Secondly, if $\psi_t = exp(tY)$ is any other other one-parameter group of contact automorphims which commutes with ϕ: $\psi_t \circ \phi = \phi \circ \psi_t$ then $\psi_t = exp(tcX)$ for some constant c. In other words, the commutator subgroup for ϕ is as small as possible , given that ϕ is part of a one-parameter subgroup. This ϕ will be constructed later.

Apply the construction of (c) of theorem 4 to obtain an engel structure $\tilde{\mathcal{D}}$ on $I\!P\xi \cong M \times S^1$ which is a deformation of the canonical engel structure and which has ϕ as its return map. In this construction, the induced contact structure on the slice $M \subset I\!P\xi$ is the original contact structure ξ on M, even after the deformation.

Now let $\Psi_t = exp(tZ)$ be any smooth one-parameter family of engel automorphisms for this deformed structure. As proved in theorem 3, Ψ_t is induced by some one-parameter family of contact automorphisms $\psi_t : (M, \xi) \to (M, \xi)$. Tracing through the constructions, we see that if the Ψ_t are to be engel automorphisms then the ψ_t must commute with the return map ϕ. Consequently $\psi_t = exp(ctX)$, for some constant c, by assumption, and the automorphisms Ψ_t must equal the automorphism induced by ψ_t via (b) and (c) of theorem 3.

It remains to construct the map ϕ with the desired property. Let M be the unit tangent bundle of a compact Riemann surface S with constant negative curvature. M has a canonical contact structure for which the geodesic flow is a one-parameter group of contact automorphisms. This contact structure on M is parallelizable, as it is for the unit tangent bundle of any oriented surface. It is well-known that in this negatively curved case the geodesic flow g_t is ergodic on M. We will see that $\phi = g_\epsilon$ for appropriate ϵ has the desired property.

If $C \subset S$ is an immersed curve then its lift to M is uniquely defined. Consequently we may safely identify a geodesic in S with the corresponding orbit of g in M obtained by following the unit tangent vector to the geodesic.

To construct ϕ recall that the closed geodesics on S form a countable collection of curves dense in S and also in M. Also recall that for each free homotopy class on S there is a unique closed geodesic. Let l_i denote the lengths of these closed geodesics. It is a countable collection of real numbers. Choose ϵ to be any real number which is smaller than the cut length of S and which is not rationally related to this collection of lengths; in other words, for which it is impossible to find a finite list of rationals $r_1, \ldots, r_N$ and lengths $l_1, \ldots, l_N$ such that $\epsilon = \Sigma r_i l_i$. The existence of such an ϵ is guaranteed in any small interval of reals because the set of such sums forms a countable set of numbers. We set $\phi = g_\epsilon$. The restriction of ϕ to any closed geodesic C is then equivalent to an irrational rotation of the circle C. It follows that if $p \in M$ lies on such a geodesic then the closure of its ϕ-orbit is all of C. The converse is also true: if $C \subset M$ is any ϕ-invariant set which is diffeomorphic to an embedded circle, then C is a closed geodesic. To see this, let $p \in C$ and let $\gamma \subset M$ be the geodesic (g-orbit) through p. Since $\phi = g_\epsilon$ we have $C \subset \gamma$. But both γ and C are immersed curves, hence they are equal.

We now check that ϕ has the desired commutator property. Suppose that $\psi_t = exp(tY)$ is a one-parameter group of diffeomorphisms which commutes with ϕ. We will show that $\psi_t = g_{ct}$ for some constant c. First observe that ψ_t must map closed ϕ-invariant embedded circles C of ϕ to other such circles. Since these Cs are precisely the closed geodesics, ϕ maps closed geodesics to closed geodesics. Since ψ_t is isotopic to the identity, $\psi_t(C)$ must be in the same free homotopy class as C, when thought of as a loop in S. Consequently $\psi_t(C) = C$ for each closed geodesic C. Since the C's are dense, it follows that the vector field Y generating ψ_t must satisfy $Y = fX$, where X is the generator of geodesic flow and f is some functon. But $\phi^* Y = Y$ since ϕ and the ψ_t commute. And $\phi^* X = X$ since $\phi = exp(\epsilon X)$. Therefore $\phi^* f = f$. Since the restriction of ϕ to each closed geodesic C is an irrational rotation, it follows that the restriction of f to each such C is a possibly C-dependent constant. But again the closed geodesics are dense and f is continuous so $f = c$, a constant, on all of M. So $\psi_t = g_{ct}$ as claimed.

QED

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