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A NONINTEGRABLE SUB-RIEMANNIAN GEODESIC FLOW ON A CARNOT GROUP

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ABSTRACT. Graded nilpotent Lie groups, or *Carnot groups*, are to sub-Riemannian geometry as Euclidean spaces are to Riemannian geometry. They are the metric tangent cones for this geometry. Hoping that the analogy between sub-Riemannian and Riemannian geometry is a strong one, one might conjecture that the sub-Riemannian geodesic flow on any Carnot group is completely integrable. We prove this conjecture to be false by showing that the sub-Riemannian geodesic flow is not algebraically completely integrable in the case of the group whose Lie algebra consists of 4 by 4 upper triangular matrices. As a corollary, we prove that the centralizer for the corresponding quadratic "quantum" Hamiltonian in the universal enveloping algebra of this Lie algebra is "as small as possible."

1. INTRODUCTION

Geometry would be in a poor state if Euclidean geodesic flow was not completely integrable – in other words, if we did not have an explicit algebraic description of straight lines in Euclidean space. Riemannian geometry, being infinitesimally Euclidean, makes frequent use of this explicit description. For example, the exponential map takes Euclidean lines through the origin to geodesics.

Sub-Riemannian geometries, also called Carnot–Caratheodory geometries, are not infinitesimally Euclidean. Rather they are, at typical points, infinitesimally modelled by Carnot groups. We will review these geometries and groups, and the relation between them momentarily. *The point of this note is to show that the Carnot geodesic flows need not be integrable. We do this by giving an example of a Carnot group of dimension* 6 *whose geodesic flow cannot be integrated by rational functions.*

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A sub-Riemannian geometry consists of a nonintegrable subbundle (distribution) V of the tangent bundle T of a manifold, together with a fiber-inner product on this bundle. These geometries arise, among other places, as the limits of Riemannian geometries. In such a limit we penalize curves for moving transverse to V so that in the limit any curve not tangent to V has infinite length. The distance between two points in a sub-Riemannian manifold is defined as in Riemannian geometry: it is the infimum of the lengths of all absolutely continuous paths connecting the two points. A theorem attributed to Rashevskii–Chow asserts that if V generates T under repeated Lie brackets, and if the manifold is connected, then this distance function is everywhere finite. Equivalently, any two points can be connected by a curve tangent to V. In this manner, every sub-Riemannian manifold becomes a metric space.

By a Carnot group we mean a simply connected Lie group G whose Lie algebra $\mathcal{G}$ is finite dimensional, nilpotent, and graded with the degree 1 part generating the algebra and endowed with an inner product. Specifically,

$$\mathcal{G} = V_1 \oplus V_2 \oplus \ldots \oplus V_r$$

as a vector space. The Lie bracket satisfies

$$[V_i, V_j] \subset V_{i+j},$$

where $V_s = 0$ for $s > r$. Also,

$$V_{i+1} = [V_1, V_i]$$

and V_1 is an inner-product space. We may think of $V = V_1$ as a left-invariant distribution on the group G. Its inner product then gives G a sub-Riemannian geometry.

Given a distribution $V \subset T$, we can, at typical points q of Q, obtain a graded nilpotent Lie algebra. In order to do this, let V also stand for the the sheaf of smooth vector fields whose values lie in V. Form

$$V^2 = [V, V],$$

$$V^3 = [V, V^2],$$

$$\vdots$$

where the brackets denote Lie brackets of vector fields. (Exercise: Show that, as sheaves: $V^j \subset V^{j+1}$.) We assume that V is bracket generating, which means that in a neighborhood of any point there is an integer r such that $V^r = T$. This is the hypothesis of Rashevskii–Chow's theorem, mentioned above.

If we evaluate the spaces V^j of vector fields at the point $q \in Q$, we obtain a flag of subspaces:

$$V_q \subset V_q^2 \subset V_q^3 \subset \ldots \subset V_q^r = T_q Q \tag{1}$$

of the tangent space at q. If the integers $\dim(V_p^j)$ are constant for p in some neighborhood of q, then q is called a *regular point*. These integers are lower semi-continuous functions of the point so that the set of regular points forms an open dense subset of Q. If these dimensions are in fact constant, then, following Gershkovich and Vershik [21], we call the integer list $(\dim V_q, \dim V_q^2, \ldots, \dim V_q^r)$ the *growth vector* or *type* of the distribution. Thus contact distributions are of type $(2n, 2n+1)$.

Associate to the filtration $V \subset V^2 \subset \ldots \subset T$ of sheaves its corresponding graded object

$$\mathrm{Gr}(V,T)_q = V \oplus V_2 \oplus \ldots V_i \ldots \oplus V_r,$$

where $V_j = V^j/V^{j-1}$ is the quotient sheaf. Because the Lie bracket of vector fields X, Y satisfies $[X, fY] = f[X,Y] \mod X, Y$ where f is a function, it induces bilinear maps: $V_j \otimes V_k \to V_{j+k}$. Putting these maps together defines, at any regular point, a Lie algebra structure on $\mathrm{Gr}_q = V(q) \oplus V_2(q) \oplus \ldots V_r(q)$. The subspace $V_q = V_1(q)$ of $\mathrm{Gr}(V,T)_q$ is the original k-plane field at that point, and Lie-generates Gr_q. It follows that Gr_q is the Lie algebra of a Carnot group G. If the distribution V comes with an inner product, then this generating subspace inherits it. Consequently G comes with a canonical left-invariant sub-Riemannian structure. This G is called the *nilpotentization* of the sub-Riemannian structure at the regular point q, as formalized by Gershkovich and Vershik ([21] and references therein).

Gromov used the idea of taking limits of a family of metric spaces to define the tangent space to any point of any metric space. (See Gromov et al. [10] and [14].) This limiting space, called the metric tangent cone, often fails to exist. For a Riemannian manifold it exists and equals the usual tangent space with its Euclidean structure. It also exists for sub-Riemannian metrics at regular points and it equals the nilpotentization G, according to a theorem of Mitchell [14]. (See also Gromov [10], [9] and Bellaiche [3]. Bellaiche also describes the tangent cone at nonregular points.)

The nilpotentization is the closest object in sub-Riemannian geometry to the Euclidean tangent space of Riemannian geometry. The match is not perfect but it is the best thing we have.

§ 1.1. The geodesic flow. A sub-Riemannian geometry can be encoded by a fiber-quadratic non-negative form $H : T^*Q \to \mathbb{R}$ on the cotangent bundle T^*Q. The kernel $\{H = 0\}$ of H is the annihilator of the distribution V. Upon polarization H becomes a bilinear non-negative form, and thus a symmetric map $g : T^*Q \to TQ$. The image of this map is V. The inner product on V is recovered through the relation $\langle g(p), v\rangle_q = p(v)$ for any

$p \in T_q^*Q$, $v \in V_q$, $q \in Q$. The Hamiltonian flow associated to H generates curves in the cotangent bundle whose projections to Q are sub-Riemannian geodesics. By a sub-Riemannian geodesic we mean a curve in Q with the property that the length of any sufficiently short subarc of the curve equals the sub-Riemannian distance between the endpoints of this arc. Such curves are necessarily tangent to V.

To write down H explicitly, pick any local orthonormal frame $\{X_1, \dots, X_k\}$ for the distribution V. Think of the X_i as fiber-linear functions on the cotangent bundle T^* so that their squares X_i^2 are fiber-quadratic functions. The Hamiltonian is then

$$H = \frac{1}{2}(X_1^2 + X_2^2 + \dots + X_k^2).$$

Remark 1. The first reference we know of the sub-Riemannian geodesic flow is Rayner [18]. This flow was subsequently investigated by Hermann [11], Brockett [3], Bailleiul [2], Gershkovich and Vershik [21], Strichartz [19], and a number of others.

Remark 2. Unlike in Riemannian geometry, there are examples of sub-Riemannian geodesics which are not the projections of these solutions to Hamilton's equations in T^*Q. See [15]–[17], [12]. But "most" geodesics are obtained as projections of these solutions.

In view of the analogies between Riemannian and sub-Riemannian geometries, the question naturally arises: *Is geodesic flow on a Carnot group always integrable*? The answer is "yes" for two step nilpotent groups. The flow on the simplest Carnot group, the Heisenberg group, has been integrated in almost every treatise on sub-Riemannian geometry. The purpose of this note is to provide an example of a three-step Carnot group whose sub-Riemannian geodesic flow is *not* integrable in terms of rational functions. Roughly speaking, this means that there is no uniform algebraic description of its "straight lines."

Let $k = \operatorname{rank}(\mathcal{D})$ and $n = \dim(Q)$. We expect the sub-Riemannian geodesic flow on T^*Q to be generically nonintegrable provided $k \geq 3$, $n \geq 6$, and the step r of the graded group Q is greater than 3. The case of rank $k = 2$ is somewhat special. The geodesic flows for the Carnot groups of type $(2,3,4)$ (the "Engel group") and $(2,3,5)$ are known to be integrable in terms of elliptic functions. See Granichina and Vershik [22], and Brockett and Dai [5]. The Carnot groups of type $(2,3,4,\dots,n-1,n)$ ("Goursat normal form") are also integrable, apparently in terms of hyperelliptic functions. Their integrability is a direct consequence of the fact that their generic coadjoint orbits have dimension 2.

In order to proceed with our nonintegrable example we need to describe how the geodesic flow for a Lie group can be pushed down to a Hamiltonian flow on the dual of its Lie algebra. We will also need to recall the definition of "complete integrability."

In the particular case where the sub-Riemannian geometry is that of a Carnot group G, then the frame X_i for $V = V_1$ can be realized by left-invariant vector fields. We may identify the space of left-invariant vector fields with the Lie algebra. Thus $X_i \in \mathcal{G}$. Then H becomes identified with a fiber-quadratic function on the dual $\mathcal{G}^*$ of the Lie algebra of G. We recall that the dual of any Lie algebra has a Poisson structure, the so-called "Lie–Poisson structure." This can be defined by insisting that

$$\{X_i, X_j\} = -[X_i, X_j],$$

where we identify elements X_i of the Lie algebra $\mathcal{G}$ with linear functions on its dual $\mathcal{G}^*$. Alternatively, if F and G are any two smooth functions on $\mathcal{G}^*$, then

$$\{F, G\}(x) = -\langle x, [dF(x), dG(x)]\rangle.$$

The Hamiltonian H induces a Hamiltonian vector-field on $\mathcal{G}^*$ by $X_H(f) = \{f, H\}$.

Geometrically, what we are doing by studying this Hamiltonian flow on $\mathcal{G}^*$ is studying the "Poisson reduction" of the sub-Riemannian geodesic flow on T^*G. The function H on T^*G is left-invariant, and, hence, so is its sub-Riemannian geodesic flow. The vector field defining this flow can then be pushed down to the quotient space $(T^*G)/G$ of the cotangent bundle by the left G action, thus defining the "Poisson-reduced" flow. Now $(T^*G)/G = \mathcal{G}^*$ in a natural way and when we push down the Hamiltonian vector field for H, we obtain the one discussed in the previous paragraph on $\mathcal{G}^*$.

We now recall the definition of completely integrable. A Hamiltonian H (or its flow) on a *symplectic* manifold of dimension $2n$ is called *completely integrable* if we can find n functions $f_1, \dots, f_n$ which are almost everywhere functionally independent ($df_1 \wedge \dots df_n \neq 0$), which Poisson-commute ($\{f_i, f_j\} = 0$), and such that H can be expressed as a function of them ($H = h(f_1, \dots, f_n)$). If the flows of the f_i are complete, then their common level sets $\{f_1 = c_1, \dots f_n = c_n\}$ are, for typical constants c_i, diffeomorphic to the quotient of $\mathbb{R}^n$ by a lattice, and on each such level, the flow of H is linear up in the covering space $\mathbb{R}^n$. The diffeomorphism is provided by the action angle coordinate.

We are interested in whether the sub-Riemannian geodesic flow on T^*G, G a Carnot group, is completely integrable. In order to proceed *we will assume that if the flow "upstairs" on T^*G is completely integrable, then so is the flow "downstairs" on $\mathcal{G}^*$*. The converse is certainly true: if the flow downstairs is integrable, then the flow upstairs is integrable. (See the paper by Fomenko and Mishchenko [13], or [1].) Our assumption is probably false in general, but we expect exceptions to be "pathological."

We need to say a few words about what we mean by "the flow downstairs being integrable." Any Hamiltonian vector field on $\mathcal{G}^*$ is necessarily tangent to the orbits of the coadjoint action of G on $\mathcal{G}^*$. The Poisson structure induces a symplectic structure on these orbits, sometimes called the Kirillov–Kostant–Souriau structure. When we say that the flow on $\mathcal{G}^*$ is "completely integrable," what we mean is that the Hamiltonian flows restricted to typical coadjoint orbits are completely integrable in the sense just described. By a typical orbit we mean one whose dimension is maximal, say $2k$. Now $2k = n - r$ where r is the rank of the Lie algebra, which is the dimension of its maximal Abelian subalgebra. In any case, complete integrability on the orbit means that there are k functionally independent functions $f_1, \ldots, f_k$ on the typical orbit which Poisson-commute with each other and such that H can be expressed in terms of them. We assume that these functions vary smoothly with the orbit. The typical orbit is defined by the vanishing of r functions $C_1, \ldots, C_r$, where the C_i form a functional basis for the coadjoint invariant functions on $\mathcal{G}^*$. These functions C_i are called Casimirs. They Poisson-commute with every function on $\mathcal{G}^*$. Pulled back to T^*G, the Casimirs form a functional basis for the bi-invariant functions on T^*G. In the nilpotent case it is known [5] that the Casimirs are rational functions. Thus, to say that the reduced system is integrable means that $H = h(f_1, \ldots, f_k; C_1, \ldots, C_r)$ for some smooth function h. We say it is *algebraically completely integrable* if the f_i and h are rational functions. (Caveat: The phrase algebraically completely integrable usually means something else in the literature, having to do with integrability in terms of Jacobians for Riemann surfaces.)

2. The example

Take G to be the group of all 4 by 4 lower triangular matrices with 1's on the diagonal. Its Lie algebra $\mathcal{N}_-$ is the space of all strictly lower triangular matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ u & y & 0 & 0 \\ w & v & z & 0 \end{pmatrix}.$$

It is generated by the three-dimensional subspace V_1 consisting of the subdiagonal matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \end{pmatrix}.$$

The functions x, y, z, u, v, w are linear coordinates on the dual of the Lie algebra, and, hence, left-invariant fiber linear functions on T^*G. The space

$V_2 = [V_1, V_1]$ is the uv plane, and $V_3 = [V_1, V_2]$ is the w-axis. For the inner product on V_1 we take the standard Euclidean one so that x, y, z correspond to the standard orthogonal coordinate system.

According to the above discussion, the sub-Riemannian Hamiltonian is

$$H = \frac{1}{2}(x^2 + y^2 + z^2).$$

The Casimirs for the Lie–Poisson structure on $\mathcal{N}_-{}^*$ are w and $uv - yw$. See [6]. Thus the typical coadjoint orbit is four-dimensional.

Theorem 1. *The geodesic flow on $\mathcal{N}_-{}^*$ generated by H is not algebraically completely integrable.*

Proof. The Kirillov–Kostant–Souriau Poisson bracket is given by the relations

$$\begin{aligned}
\{z, u\} &= w, \\
\{v, x\} &= w, \\
\{y, x\} &= u, \\
\{z, y\} &= v
\end{aligned}$$

with all the other Poisson brackets of the coordinate functions equalling zero. We choose x, z, $\tilde{u} = \dfrac{u}{w}$, and $\tilde{v} = \dfrac{v}{w}$ as Darboux coordinates on a generic orbit. "Generic" means that $w = w_0 \neq 0$ and $uv - yw = C \neq 0$. One easily checks that x, z, $\tilde{u}$, and $\tilde{v}$ are independent coordinates on a generic orbit and that

$$\begin{aligned}
&\{z, \tilde{u}\} = 1, \\
&\{\tilde{v}, x\} = 1, \\
&\{z, x\} = \{z, \tilde{v}\} = \{x, \tilde{u}\} = \{\tilde{u}, \tilde{v}\} = 0.
\end{aligned}$$

In these coordinates the Hamiltonian has a form

$$H(x, z, \tilde{u}, \tilde{v}) = x^2 + z^2 + \left(\frac{C - uv}{w_0}\right)^2 = x^2 + z^2 + \left(\frac{C}{w_0} - w_0 \tilde{u}\tilde{v}\right)^2.$$

Under the following linear symplectic change of coordinates

$$\begin{aligned}
x &= \sqrt[3]{w_0}\,\hat{x}, \\
z &= \sqrt[3]{w_0}\,\hat{z}, \\
\tilde{u} &= \frac{\hat{u}}{\sqrt[3]{w_0}}, \\
\tilde{v} &= \frac{\hat{v}}{\sqrt[3]{w_0}},
\end{aligned}$$

the Hamiltonian becomes $w_0^{\frac{2}{3}}(\hat{x}^2 + \hat{z}^2 - 2C\sqrt[3]{w_0}\hat{u}\hat{v} + \hat{u}^2\hat{v}^2 + K)$, where K is some constant depending on w_0 and C only. This Hamiltonian is proportional to the famous Yang–Mills Hamiltonian, which Ziglin proved to be rationally nonintegrable. See Ziglin [23], [24]. □

3. Quantization: Extension to the universal enveloping algebra

The nonintegrability in rational functions of the system considered above has some purely algebraic consequences.

Lemma 1. *Let $\mathcal{N}_-$ be the algebra of nilpotent lower triangular 4×4 matrices and $\mathrm{Pol}(\mathcal{N}_-{}^*)$ be the algebra of polynomials on its dual space $\mathcal{N}_-{}^*$. Let $\{F, H\} = 0$, where $H = \frac{1}{2}(x^2 + y^2 + z^2)$ and $F \in \mathrm{Pol}(\mathcal{N}_-{}^*)$. Then $F = P(H, w, uv - yw)$. Here x, y, z, u, v, w were defined in the previous section and P is a polynomial.*

Proof. Taking into account the fact that the dimension of the generic orbit of the coadjoint representation in $\mathcal{N}_-{}^*$ is 4, we see that if F commuted with H but were not of the form $P(H, w, uv - yw)$, then the system defined by H would be completely integrable. But this contradicts Theorem 1. □

Given any function f in $\mathrm{Pol}(\mathcal{N}_-{}^*)$, its centralizer with respect to Poisson bracket always contains the polynomials $F(f, w, uv - yw)$. For, as mentioned earlier, w, $uv - yw$ are the *Casimirs* for $\mathcal{N}_-{}^*$: they generate the center of $\mathrm{Pol}(\mathcal{N}_-{}^*)$. Thus the lemma asserts that the centralizer of H is as small as possible.

We will finish off by proving a similar result for the universal enveloping algebra $U(\mathcal{N}_-)$. $U(\mathcal{N}_-)$ can be thought of as the algebra of left-invariant differential operators on the Lie group N of upper triangular matrices with 1's on the diagonal (or on certain homogeneous spaces for N). It is generated as an algebra over R by $\mathcal{N}_-$ (the 1st order differential operators) and the unit 1 (the identity operator). Let E_{ij} be the standard unit matrix with only nonzero (i, j) entry equal to 1. Set $X = E_{21}$, $Y = E_{32}$, $Z = E_{43}$, $U = E_{31}$, $V = E_{42}$, $W = E_{41}$. Then X, Y, Z, U, V, W together with 1 generate $U(\mathcal{N}_-)$. Observe that this notation is consistent with that used for the elements x, y, z, u, v, w for $\mathrm{Pol}(\mathcal{N}_-{}^*)$. Thus w is a linear function on $\mathcal{N}_-{}^*$, which is to say an element of $\mathcal{N}_-$.

Let $\widetilde{H} = \frac{1}{2}(X^2 + Y^2 + Z^2) \in U(\mathcal{N}_-)$. It is the "quantization" of our $H \in \mathrm{Pol}(\mathcal{N}_-{}^*)$.

By a theorem of Hormander, it is a hypoelliptic differential operator. This is almost as good as being elliptic. It is well known that the center of $U(\mathcal{N}_-)$ is generated by W and $UV - YW$. (See Dixmier [7].) Consequently if R is any element of $U(\mathcal{N}_-)$, then its commutator algebra *contains* the subalgebra generated by R, W and $UV - YW$.

Theorem 2. *Any element F in $U(\mathcal{N}_-)$ which commutes with $\widetilde{H}$ is of the form $F = P(\widetilde{H}, W, UV - YW)$ for some polynomial P.*

There is no ordering problem in defining the element $P(H, W, UV - YW)$ since W and $UV - YW$ commute with everything. The theorem asserts that $\widetilde{H}$ commutes only with those elements which every operator must commute with. It suggests that $\widetilde{H}$, as an operator, should exhibit "quantum chaos."

This theorem is a special case of a result which holds for any finite-dimensional Lie algebra $\mathcal{G}$. The result may be well known to experts, but we will present it here in any case.

Let $U(\mathcal{G})$ be the universal enveloping algebra of the finite-dimensional Lie algebra $\mathcal{G}$ and $Z(G) \subset U(G)$ its center. $U(\mathcal{G})$ is filtered by degree. An element is said to have degree less than or equal to k if it is a sum of monomials of the form $X_1 X_2 \dots X_s$ with the $X_i \in \mathcal{G}$ and $s \leq k$. If $s = k$ for one of these monomial terms, then its degree equals k. The corresponding graded algebra $\mathrm{Gr}(U(G))$ is canonically isomorphic to the algebra $\mathrm{Pol}(\mathcal{G}^*)$ of polynomials on $\mathcal{G}^*$. The operator bracket respects the filtration so that it induces a Lie bracket on $\mathrm{Pol}(\mathcal{G}^*)$. This is of course the KKS Poisson bracket $\{,\cdot,\cdot\}$ on $\mathcal{G}^*$.

Let $U(\mathcal{G})_k$ denote the subspace of elements of degree k or less. The quotient map $\sigma_k : U(\mathcal{G})_k \to U(\mathcal{G})_k / U(\mathcal{G})_{k-1} \cong \mathrm{Pol}(\mathcal{G}^*)_k$ takes elements of degree k to homogeneous polynomials of degree k. If an element $\tilde{F} \in U(G)$ has degree k, then we call $\sigma_k(F)$ its principal symbol. If two elements in $U(\mathcal{G})$ commute, then their principal symbols must Poisson-commute in $\mathrm{Pol}(\mathcal{G})^*$. This follows directly from the relation between the operator and Poisson brackets.

There is a symmetrization map $\phi : \mathrm{Pol}(\mathcal{G}^*) \to U(\mathcal{G})$ which is a kind of inverse to the symbol maps. It is a linear isomorphism but of course not an algebra homomorphism. When restricted to the subspace of homogeneous polynomials of degree k, it satisfies $\sigma_k \circ \phi = \mathrm{Id}$.

The center $Z(\mathcal{G})$ of $U(\mathcal{G})$ is finitely generated by elements $\tilde{f}_1, \dots, \tilde{f}_r$. These elements may be chosen so that their principal symbols $f_1, \dots, f_r$ generate the center of $\mathrm{Pol}(\mathcal{G})$ and so that $\tilde{f}_i = \phi(f_i)$. (See Dixmier [7], or Varadarajan [20], Theorem 3.3.8, p. 183.) Elements of either center are called *Casimirs*. (If $\mathcal{G}$ is semi-simple, then the number r of Casimirs is the rank of the Lie algebra.)

Proposition 1. *Let $\tilde{H}$ be an element of $U(\mathcal{G})$ of degree m and $H = \sigma_m(\tilde{H})$ its principal symbol. Suppose that the commutator algebra of H in $\mathrm{Pol}(\mathcal{G}^*)$ is generated by the Casimirs $f_1, \dots, f_r$ together with H. Then the commutator algebra of $\tilde{H}$ in $U(\mathcal{G})$ is generated by the Casimirs $\tilde{f}_1, \dots, \tilde{f}_m$ together with $\tilde{H}$.*

In other words, if the commutator of the principal symbol is as small as possible, then the same is true for its quantization $\tilde{H}$.

Proof. Suppose $\tilde{F}$ commutes with $\tilde{H}$. Let k denote the degree of $\tilde{F}$ and set $F = \sigma_k(\tilde{F})$. As discussed above, F Poisson-commutes with H. By hypothesis $F = p(f_1, \dots, f_r, H)$ for some polynomial p. Since the $\tilde{f}_i$ are in the center of $U(G)$, the element $\tilde{p} = p(\tilde{f}_1, \dots \tilde{f}_r, \tilde{H})$ is a well-defined element of $U(\mathcal{G})$, independent of the ordering of the factors $\tilde{f}_i, \tilde{H}$. Clearly $\tilde{p}$ commutes with $\tilde{H}$. Moreover $\sigma_k(\tilde{p}) = F$ so that $\sigma_k(\tilde{F} - \tilde{p}) = 0$. It follows that $\tilde{F} - \tilde{p} = \tilde{F}_2$ is an element whose degree is $k-1$ or less which commutes with $\tilde{H}$.

Repeating this argument with $\tilde{F}_2$ in place of $\tilde{F}$, we obtain a polynomial p_2 in $f_1, \dots, f_r$, H and a corresponding element $\tilde{p}_2$ in $U(\mathcal{G})$ such that $\tilde{F} - (\tilde{p} + \tilde{p}_2)$ commutes with $\tilde{H}$ and has degree $k-2$ or less. Continuing in this fashion, we eventually descend to degree 0, in which case

$$\tilde{F} = \tilde{p} + \tilde{p}_2 + \dots + \tilde{p}_{k-1}$$

is a polynomial in the $\tilde{f}_i, \tilde{H}$ as claimed. □

4. Remaining problems

1. Show that the flow on $\mathcal{N}_-{}^*$ is not integrable in terms of general smooth functions. This appears to require one to obtain more detailed knowledge of the dynamics, as the Ziglin method fails.

2. Show that the full flow on T^*G in not integrable, either rationally or in the full sense. See the next-to-last paragraph of Subsec. 1.1 for difficulties involved here.

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