BLOW-UP, HOMOTOPY AND EXISTENCE FOR PERIODIC SOLUTIONS OF THE PLANAR THREE-BODY PROBLEM

Abstract. Deleting collisions from the configuration space of the planar N-body problem yields a space with a large interesting set of free homotopy classes of loops, classes which are encoded by "syzygy sequences" when $N = 3$. This expository piece centers on the question " Is every free homotopy class of loops realized by a periodic solution to the problem?" We report on the recent affirmative answer [30] for the case of non-zero but small angular momentum and three equal or near-equal masses. The key tool is the McGehee blow-up [20] as implemented by Rick Moeckel in the 1980s. After recounting some history and motivation, about a third of this article exposes the blow-up method. We use an energy balance under scaling transformations to motivate McGehee's blow-up transformation. We give an explicit description of the blown-up and reduced phase space for the planar N-body problem, $N \geq 3$ as a complex vector bundle over $[0, \infty) \times \mathbb{CP}^{N-2}$. We end by returning to the angular momentum zero case where we conjecture the answer is 'no'. We support this conjecture by recent work of Connor Jackman [17] and of Danya Rose [40].

1. Introduction

The following theorem inspired much of my work on the N-body problem.

Background Theorem 1. *Let (M, ds^2) be a compact Riemannian manifold. Then every free homotopy class of loops on M is realized by a closed geodesic.*

If one continuous loop in M can be continuously deformed into another without leaving M then we say that the two loops are "freely homotopic" [1]. Free homotopy

[1] For those familiar with the fundamental group, we emphasize that the adjective "free" means that there is no fixed base point through which all loops must pass. The space of free homotopy classes typically does not form a group, rather it is isomorphic to the set of conjugacy classes of the fundamental group.

defines an equivalence relation on loops in M. The resulting equivalence classes are the free homotopy classes of loops.

To move from Riemannian geometry to the planar three-body problem we replace the geodesic equations by Newton's equations, and the Riemannian manifold of the above theorem by the configuration space for the planar three-body problem. This configuration space is the product of three copies of the Euclidean plane. Newton's equations have singularities along the collision variety where two or more of the bodies collide. Excluding the collision variety from configuration space induces a large space of free homotopy classes where previously there were none.

Open Question. 1. *Is every free homotopy class for the planar Newtonian three-body problem realized by a collision-free periodic solution?*

We address a reduced version of this question , where "reduced" means modulo the group G of rigid motions of the plane, which is the built-in group of symmetries of Newton's equations. Consequently we can "reduce" Newton's equations to obtain equations on the quotient of the three-body configuration space by G. We call this quotient space *shape space* since its points represent oriented congruence classes of triangles. See [34] for details and a derivation of the structure of shape space. Shape space is diffeomorphic to $\mathbb{R}^3$. Under this diffeomorphism the collision variety (modulo G) becomes three rays issuing forth from the origin. The free homotopy classes of shape space minus collisions are called "reduced free homotopy classes". Momentarily we describe how to encode reduced free homotopy classes in a simple combinatorial way.

We also do not require our solutions are periodic, but rather that they are "reduced periodic" meaning periodic modulo G. Concretely, if r_{ij} denotes the interparticle distances, i.e. the sides of the triangle formed by the three bodies, then we insist that these distances for our solutions satisfy $r_{ij}(t+T) = r_{ij}(t)$ where T is the reduced period [2] .

[2]Strictly speaking, this T may actually be half the reduced period. For example, if after time T the inital and final triangle are related by a reflection, then after time $2T$ the two triangles are the same.

The reduced free homotopy classes are conveniently encoded in the astronomical language of "syzygies". A syzygy is an instant or configuration for which the three bodies lie in a line. Non-collision syzygies are marked 1,2, or 3, depending upon which of the three masses lies in the middle at the syzygy instant. Consider a curve c in the configuration space of the three-body problem which is closed modulo rotation. Write out its syzygy sequence on the circle. We get a periodic list of 1's 2's and 3's. The list is subject to the "non-stuttering" cancelation rule: any time we see a 11, a 22 or a 33 we delete it. We call the resulting sequence the "reduced syzygy sequence" of the free homotopy class. For example 32 is the reduced syzygy sequence of 12112321 since $12112321 = 122321 = 1321 = 32$ where the last cancellation arises because the word is written on the circle. It can be proved that two reduced-periodic collision-free curves represent the same reduced free homotopy class if and only if their reduced syzygy sequences are equal.

Theorem 1 ($(RM)^2$ [30])**.** *For equal or near-equal masses, and angular momenta sufficiently small but nonzero, every reduced syzygy sequence, and thus every reduced free homotopy class for the planar three-body problem is realized by a reduced periodic orbit for the Newtonian planar three-body problem.*

1.1. **My path through the Variational Wilderness.** "Look at every path closely and deliberately. Try it as many times as you think necessary. Then ask yourself, and yourself alone, one question. [...] Does this path have a heart? "

–from the preface of the book "Keep the River on your Right", by Tobias Schneebaum; a slight variation on a well-known passage in Carlos Castaneda's "The Teachings of Don Juan: A Yaqui Way of Knowledge"

The variational proof of the background theorem, theorem 1, proceeds as follows. Fix a free homotopy class. Define m to the infimum of the lengths of all loops which realize this class. Choose a minimizing sequence : a sequence of loops in the class whose lengths tend to m. Because M is compact, the Arzela-Ascoli theorem guarantees that the sequence has a convergent subsequence. Call c_* the loop to which the subsequence converges. Standard methods from the calculus of variations now show that c_* lies in the class and its length m, proving the theorem.

The proof just sketched provides an archetypal example of the *direct method of the calculus of variations* in action. I learned it, and loved it, in grad school. But for me, back in grad school, Celestial Mechanics was the land of old famous long-dead men, a world of very hard problems of no real interest. I was certain I would never work in it. I studiously avoided the entire last part of the book by Abraham-Marsden, since that is the part titled 'Celestial Mechanics'.

Instead, coming out of grad school, I tried my hand in what was in vogue, in gauge theory, symplectic reduction, and eventually I was led into problems in subRiemannian geometry and optimal control where I probably did my first real serious piece of work. I was led into subRiemannian geometry through the work of Wilczek and Shapere who had shown that the problem faced by a falling cat when trying to right herself when dropped from upside down with no angular momentum can be viewed as a kind of optimal control problem mixed with gauge theory. I kept simplifying the cat problem until she consisted of three mass points. At this juncture, I knew I was perilously close to working on the three-body problem but I studiously avoided actually working on it. In 1995 or 1996 Alan Weinstein, a mentor of mine in grad school, told me that Wu-yi Hsiang had been looking into the three-body problem from a perspective very similar to mine: variational, combined with equivariant differential geometry. Hsiang and I got together in a cafe on Euclid Avenue in Berkeley one afternoon and spent perhaps three hours together. His personality is a force of nature. This force and Hsiang's optimism and enthusiasm convinced me to begin work on some baby problems within the three body problem. During that visit, Hsiang posed the reduced version of question 1 , and drew pictures on yellow pads of paper which were reminescent of figures 4 and 5 of this paper.

Very soon after my encounter with Hsiang, I started a seminar at UCSC on the N-body problem. Chris Golé, who had written a wonderful book on Symplectic Twist Maps, was visiting UCSC as an assistant professor and attended regularly. Bill Burke , one of my few real friends on the UCSC faculty, and a physics professor at UCSC also attended. (Bill would die a few years later at 52 when his pickup truck was flipped over in a high wind in Hurricane, Utah, as he was driving home from a Grand Canyon rafting trip.) A month in to our seminar, and Chris Golé took me aside and told me "Richard, if you are serious about doing work in the N-body problem then you must go to Paris. You have to visit the Bureau des Longitudes and talk with Alain Albouy, Alain Chenciner and Jacques Laskar. They do phenomenal work in mathematical celestial mechanics."

The next year I had a sabbatical 1997-1998, and my family and I took that sabbatical year at CIMAT, in Guanajuato, Mexico. While there I began my book on SubRiemannian Geometry and also invited myself to Paris in the Spring. I stayed in Paris for 6 weeks that first time and became lifelong friends with the two Alains and with Jacques.

My conversations with Albouy and Chenciner and eventual collaboration with Chenciner began slowly, and evolved out of their incredibly careful, thorough, and exacting referee work of some of my papers for the journal "Nonlinearity". I will not repeat the story of how, at the end of the millenium, in December of 1999, Chenciner and I rediscovered the remarkable figure eight orbit of Cris Moore. (You can find a version of that story on my web page, for example.) Upon seeing an early draft of our paper, Phil Holmes told us that his student C Moore had done numerical based on a similar work six years before us [35]. And Robert MacKay pointed Chnenciner to an amazing paper of Poincaré, over a century earlier in which Poincaré [37] used the direct method to answer a variant of question 1. He proved the existence of a reduced periodic solution in almost every reduced *homology* class, provided we replace the Newtonian $1/r^2$ force law by a $1/r^3$ "strong-force" law. [3]

[3]For a $1/r^a$ potential, so $1/r^{a+1}$ force law, the action of an orbit segment with an isolated collision is finite if and only if $a < 2$.

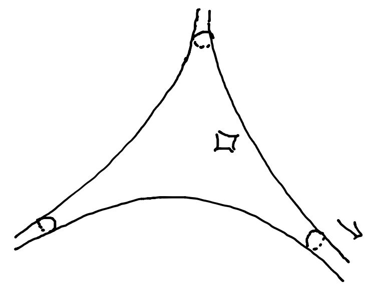


FIGURE 1. Slip the circles off the ends....

In addition to his beautiful 3 page paper, I recommend [9] or [5] for history and details of successes and failures of the variational method applied to the N-body problem.

The main challenge in getting the direct method of the calculus of variations to work in a three-body problem is the non-compactness of its configuration space. Minimizing sequences of loops may leave the configuration space, escaping 'to infinity', in their attempts to minimize. See figure 1.1 The most challenging escape to infinity to prevent is not the ultraviolet escape (distances between bodies tending to infinity) but rather the infrared escape : distances between bodies going to zero, which is to say the possibility that a minimizing sequence of paths converges to a path in which two or more of the bodies suffer a collision.

After the figure eight work, I continued trying to use variational methods and geometric perspectives to establish new results for the Newtonian N-body problem. Our "choreographic" success, my new French, Catalan, and New Jersey friends [12], and the surprising simplicity and beauty of the variational proof of theorem 1 kept me on a 17 year long path through the underbrush of the calculus of variations in trying to establish some version of Theorem 1. Throughout that time, I worked almost exclusively in the case of zero angular momentum. The reason I insisted on restricting to zero angular momentum is that the restriction arises in an extremely natural way out of the map from configuration space tothrough shape space. One realizes this projection to be a *Riemannian submersion* and that being orthogonal to the fibers (the rotational orbits of a single triangle) is equivalent to having zero angular momentum, and minimizers between orbits must have zero angular momentum. In physical terms, if you fix a curve on shape space, and minimize the length of all realizations of this shape curve by curves in configuration space, then the resulting minimizers are *exactly* those curves which project onto the given curve **and** having angular momentum zero.

In retrospect, the long variationally based path I took was a path with a heart. But it did not bring me closer to a resolution of Question 1.

1.2. **Breakthrough.** I had a sabbatical in Portland in the Spring of 2014. I had largely given up on making progress on Question 1 and decided that it was time that I performed some numerical experiments to get some idea of whether the answer was yes or no. To do this, I set myself the task of sorting through orbit segments for the zero-angular-momentum-equal-mass-three-body problem, by numerically integrating a large variety of initial conditions up until they generated a syzygy sequence of some fixed length N. I planned to take this as “raw data” and look for gaps – certain nonoccurences of subwords – in the resulting length N words. I figured I could get up to words of length N=10, by shooting from the collinear plane in shape space. I rediscovered for the fourth or fifth time that programming and numerical analysis are not paths with a heart for me. Within a month I could see that I would need more than a year to complete my appointed task. I would need help! At first I tried to enlist the aid of my old kayak friend and life-long Fortran programmer Michael Schlax who was living nearby in Corvalis. But it soon became clear that this would be too slow- -Michael was a geophysicist and statistician by training, not a mathematician. And I was not enjoying learning Fortran. So I decided to seek out my old friend Carles Simó and get one of his students, or even better, him, to aid me in my numerical searches. Carles Simo has a reputation as one of the most inventive, careful numerical analysts working in celestial mechanics. He is also a mathematician and a friend. So we can talk. My subRiemannian book had gotten me invited to a trimester at IHP in Paris for the Fall quarter of 2014 and I took advantage of that trip to invite myself to spend time with Carles in in Barcelona. The first afternoon I first met with CarlCarles es and explained Question 1. He looked at me with his piercing eyes and asked "Richard, why do you care?"

I had been working on this problem 17 years. Carles' s question was a stab to my heart! It knocked the breath out of me. But I knew Carles did not mean to hurt – he is simply a direct man who does not waste time or mince words. So I tried to explain why I cared. Carles listened. The next morning when we met again, he began "Richard, if what you think is true about this Question, (that all free homotopy classes are realized) then there has to be a dynamical mechanism."

With those few words I switched paths! I abandoned the variational path, and asked myself what "dynamical mechanisms" do I know? What mechanisms which work for general Hamiltonian systems? I realized I knew only two "dynamical mechanisms" : that of KAM torii and that hyperbolic tangles. KAM would be of no help. But Moser, in his famous book [36], had shown clearly how tangles yield symbolic dynamics in a celestial mechanics problem, the Sitnikov problem.

I wracked my brain. Who had done similar work, but for the full three-body problem? Rick Moeckel! And Rick had been my main colloborator these last three years! I reread some of Rick's papers [([27], [23], and [29])] from the 1980s. I discovered that back then he had essentially solved my problem! A few small gaps remained to fill, but Moeckel had done the huge bulk of the work nearly 30 years earlier.

The goal of the remainder of this paper is to explain what Rick did, how we used it , and try to motivate his work and in particular the McGehee blow-up.

2. Background: Equations and Solutions.

2.1. **Equations.** The classical three-body problem asks that we solve the system of non-linear ODEs:

$$\begin{aligned} m_1\ddot{q}_1 = & \quad F_{21} + F_{31} \\ m_2\ddot{q}_2 = & \quad F_{12} + F_{32} \\ m_3\ddot{q}_3 = & \quad F_{23} + F_{13}. \end{aligned} \tag{1}$$

where

$$F_{ab} = Gm_a m_b \frac{q_a - q_b}{r_{ab}^3} \tag{2}$$

is the force exerted by mass m_a on mass m_b and

$$r_{ab} = |q_a - q_b|, q_a \in \mathbb{R}^d, \text{ and } m_a, G > 0.$$

Here $a, b = 1, 2, 3$ label the bodies. The dimension d for us will be 2. The standard value is $d = 3$.) The m_a represent the values of point masses whose instantaneous positions are $q_a(t)$. The double dots indicate two time derivatives: $\ddot{q} = \frac{d^2q}{dt^2}$. The constant G is Newton's gravitational constant and is physically needed to make dimensions match up. Being mathematicians, we can and do set $G = 1$.

2.2. **Solutions of Euler and Lagrange.** The only solutions to the three-body problem for which we have explicit formulae were found by Euler [14] and Lagrange [19] in the last half of the 18th century. See figures 2, 3. Their solutions are central to our story.

For Lagrange's solution, place the three masses at the vertices of an equilateral triangle and drop them: let them go from rest. They shrink homothetically towards their common center of mass, remaining equilateral at each instant. The solution ends in finite time in triple collision. This motion forms half of Lagrange's triple collision solution. To obtain the other half of Lagrange's solution use time-reversal invariance to continue this solution backwards in time. In the full solution the three masses explode out of triple collision, reach a maximum size at the instant at which we dropped the three masses, and then shrink back to triple collision, staying equilateral throughout. A surprise is that the Lagrange solution works regardless of the mass ratios $m_1 : m_2 : m_3$.

For Euler's solutions, place the masses on the line in a certain order: $q_i < q_j < q_k$ so as to form a special ratio $q_k - q_j : q_j - q_i$. (This special ratio depends on the mass ratios and also the choice of mass m_j on the middle and is the root of a fifth degree polynomial whose coefficients depend on the masses.) Again drop them. They stay on the line as they evolve and again the similarity class of the (degenerate) triangle stays constant: this ratio of side lengths stays constant. (In case the two masses at the ends are equal then the special ratio is $1 : 1$: place m_j at the midpoint of m_i and m_k.)

The solutions described are part of a family of explicit solutions. For every one of the solutions in these families the similarity class formed by the three masses stays constant in time during the evolution. Each mass moves on its own Keplerian conic with the center of mass of the triple as focus, the solutions described above being the special case of degenerate (colinear) ellipses. We derive these families analytically in section 4.3.1 below.

All together these solutions form five familes. The corresponding shapes are called "central configurations". The Lagrange solutions count as two, one shape for each orientation of a labelled equilateral triangle. The Euler solutions count as three, one for each choice of mass in the middle.

For almost all (Newtonian) time the solutions of theorem 1 are very close to one of the three Euler solutions. The Lagrange solutions act as bridges between various Eulers.

3. Shape sphere. Blow-up and reduction, first pass.

A basic aid to understanding the planar three-body problem is the *shape sphere*, a two-sphere whose points represent oriented similarity classes of triangles. At

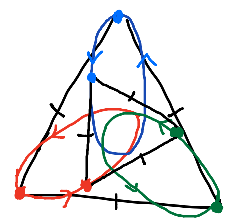


FIGURE 2. A Lagrange Solution.

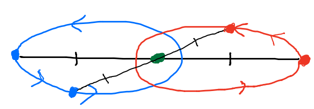


FIGURE 3. An Euler Solution.

each instant of time the three bodies form the vertices of a triangle. Call two triangles "oriented similar" if one can be brought to the other by a composition of translations, rotations, and scalings. The resulting space of equivalence classes forms the shape sphere. See figure 4. This sphere has 8 marked points, the 5 central configurations just described L_+, L_-, E_1, E_2, E_3 and The 3 binary collision points labelled B_{12}, B_{23}, B_{31} . The sphere's equator represents the space of collinear triangles. The 3 binary collision points, and 3 Euler central configurations lie on this equator, interleaved so as to be alternating.

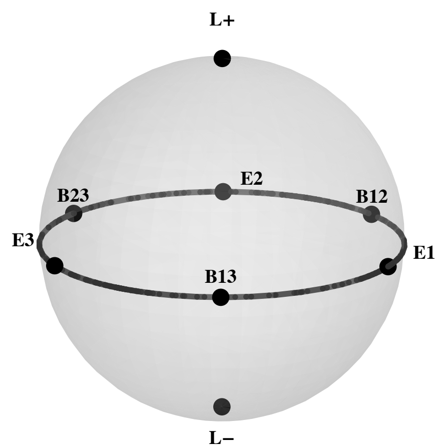


FIGURE 4. The shape sphere. Lagrange points, Euler points, and collision points marked. The equator consists of collinear triangles. figure courtesy of Rick Moeckel

The earliest occuring picture of the shape sphere in the context of celestial mechanics with which I am familiar is [24] . You will find a detailed exposition of the shape sphere and its relation to the three-body problem in [34].

We summarize how the shape sphere arises out of the three-body problem. The configuration space for the three-body problem, with collisions allowed, is $\mathbb{C}^3$ with a point $q = (q_1, q_2, q_3) \in \mathbb{C}^3$ representing the 3 vertices of the triangle - the positions of the 3 bodies. We have identified $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way: $x + iy \in \mathbb{C}$ corresponds $(x, y) \in \mathbb{R}^2$. A standard trick from Freshman physics allows us to restrict the problem to the center-of-mass zero subspace:

$$\mathbb{E}_{cm} = \{q \in \mathbb{C}^3 : m_1 q_1 + m_2 q_2 + m_3 q_3 = 0\} \cong \mathbb{C}^2 \subset \mathbb{C}^3.$$

(See the beginning of section 5 below.) In $\mathbb{E}_{cm}$ the binary collision locus become three complex lines which intersect at the origin 0. The origin represents triple collision. The masses endow $\mathbb{C}^3$ with a canonical metric called the "mass-metric" (eq. (4)) and relative to that metric the distance from triple collision is given by r where

$$r^2 = m_1|q_1|^2 + m_2|q_2|^2 + m_3|q_3|^2.$$

(See eq. (8).) Take the sphere

$$\{r = 1\} := S^3 \subset \mathbb{C}^2 \cong \mathbb{E}_{cm}.$$

Because the three-body equations are invariant under rotations they descend to ODEs on the quotient of $\mathbb{C}^2 = \mathbb{E}_{cm}$ by the group S^1 of rotations. This quotient space is topologically an $\mathbb{R}^3$. *We call this* $\mathbb{R}^3$ *"shape space".* To understand this quotient note that the rotation action leaves r unchanged but moves points on S^3 around according to $(Z_1, Z_2) \mapsto (uZ_1, uZ_2)$, $u \in S^1 \subset \mathbb{C}$. (Here Z_1, Z_2 are any complex linear coordinates for $\mathbb{E}_{cm}$.) This is the circle action used to form the Hopf fibration:

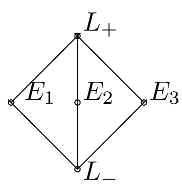
$$Hopf : S^3 \to S^3/S^1 = S^2 = \text{ shape sphere }.$$

Points of the quotient $\mathbb{R}^3$ represent oriented congruence classes of triangles: planar triangles modulo translation and rotation, but not scaling. Express $\mathbb{R}^3$ in spherical coordinates $(r, s), s \in S^2$. Then the origin $r = 0$ corresponds to triple collision. A point s on the sphere represents a ray $rs, r \geq 0$ of triangles all having the same shape. The collision locus $\mathcal{C} = \{r_{12} = 0 \text{ or } r_{23} = 0 \text{ or } r_{31} = 0\}$ is represented by the three rays corresponding to the three binary collision points $B_{12}, B_{23}, B_{31} \in S^2$.

Newton's equations break down at triple collision $r = 0$. McGehee blow-up is a change of variables (equations (12)) which converts Newton's equations to a system of ODEs which is well-defined when $r = 0$. The locus $r = 0$ in the new variables is called "the collision manifold" and forms a bundle over the shape sphere. The blown-up system of ODEs has exactly 10 fixed points, all on the collision manifold, a pair of fixed points lying over each of the five central configurations. For a chosen central configuration, one element of the pair corresponds to the homothetic arc incoming to triple collision, as in our original description of the Lagrange solution, while the other element of the pair corresponds to the initial segment of that solution which explodes out from triple collision.

The 10 fixed points on the collision manifold have stable and unstable manifolds, parts of which stick out of the collision manifold, and which intersect in complicated ways, as per the Smale Horseshoe and heteroclinic tangles. See figure 6. Moeckel investigated these manifolds and their relations in seminal works [28], [23], [25], [24], [27], and [29] where he proved existence of "topological heteroclinic tangles" between them. Simó and Suslin had also proved existence of connections between the various collision manifolds with careful numerical evidence in **??**.

One finds the following abstract graph



in several of these papers ([27], p. 53, Theorem 1′. Figure 2 of [29] becomes our graph after deleting the vertices labelled with B's and s edges incident to these B's.) Moeckel's theorem in [27], based on the intersections between stable and unstable manifolds of the 10 fixed points, asserts that all paths in this graph are "realized" by solutions to the three-body problem provided the angular momentum, energy and masses are as per theorem 1. Embed this graph in the shape sphere as indicated by figure 5. Call the embedded graph the "concrete connection graph".

The dynamical relevance of the concrete connection graph has to do with the Isosceles three-body problem. When two of the masses are equal, say $m_1 = m_2$, then the isosceles triangles $r_{13} = r_{23}$ form an invariant submanifold of the three-body problem whose dynamics is called the "Isosceles three-body problem". These 2-3 Isosceles triangles form a great circle in shape space which passes through both Lagrange points, the binary point B_{23} , and the Euler point E_1. If all three masses are equal we have have three Isosceles subproblems represented by three great circles on the shape sphere. Take one-half of each great circle, namely that half whose endpoints are the two Lagrange points and which contains the Euler point. In this way we form the concrete connection graph whose edges are Isosceles semi-circles.

Observe that the shape sphere minus the three binary collision points retracts onto the concrete connection graph. Theorem 1 follows immediately from this observation and Moeckel's theorem refered to above, once we know that the realizing solutions of Moeckel's theorem, projected onto the shape sphere, stay C^0-close to corresponding edges in the concrete connection graph. For details see section 6 of this article or [30].

4. Metric Set-up. McGehee Blow-up

It is no more work to perform the blow-up for the N-body problem in d-dimensional Euclidean space, rather than our special case of the three-body problem in the plane. The d-dimensional N-body equations are:

$$m_a \ddot{q}_a = \Sigma_{b \neq a} F_{ba} \qquad , q_a \in \mathbb{R}^d \tag{3}$$

with the forces F_{ba} as above.

4.1. **Metric Reformulation.** Let

$$\mathbb{E} = (\mathbb{R}^d)^N$$

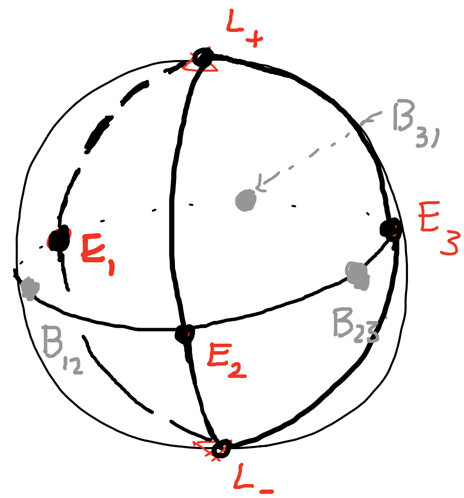


FIGURE 5. The concrete graph, embedded in the shape sphere.

denote the N-body configuration space. Write points of $\mathbb{E}$ as $q = (q_1, \ldots, q_N)$ and think of the points as the N-gons in d-space. The masses endow $\mathbb{E}$ with an inner product,

$$\langle q, v \rangle = \Sigma m_a q_a \cdot v_a \tag{4}$$

called the *mass inner product.* Here $\cdot$ denotes the standard inner product on $\mathbb{R}^d$. Then the standard kinetic energy is

$$K = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle. \tag{5}$$

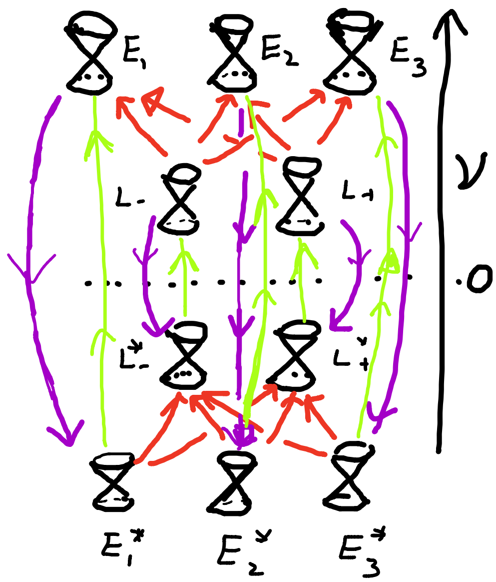


FIGURE 6. The equilibria arising upon blow-up and relations between their stable and unstable manifolds. The purple and green arrows are the rest cycles described in figure 5.

Let ∇ be the gradient associated to this metric: $df_q(v) = \langle \nabla f(q), v \rangle$, so that $(\nabla f)_a = \frac{1}{m_a} \frac{\partial f}{\partial \vec{q}_a}$. Then the N-body equations take the simple form

$$\ddot{q} = \nabla U(q) \tag{6}$$

where U is the *negative* of the standard potential V:

$$U = -V = \Sigma_{a<b} \frac{m_a m_b}{r_{ab}}, \tag{7}$$

the sum being over all distinct pairs a, b. As is well known, the total energy

$$H(q, \dot{q}) = K(\dot{q}) - U(q) = K(\dot{q}) + V(q).$$

is conserved, where "conserved" means constant along solutions. (Another synonym for "conserved quantity" is "constant of the motion".) We use

$$r = \sqrt{\langle q, q \rangle} \tag{8}$$

to measure the size of our configurations. Lagrange proved that

$$r^2 = \Sigma_{a<b} m_a m_b r_{ab}^2 / \Sigma m_a$$

provided we are in center of mass coordinates: $\Sigma m_a q_a = 0$. Then

$$r = 0 \Leftrightarrow \text{total collision: all masses coincide}$$

while

$$U = \infty \Leftrightarrow \text{some collision: some pair of masses coincide.}$$

Exercise 1. *Use the metric reformulation of Newton's equations eq (3), the fact that U is homogeneous of degree -1 and Euler's identity for homogeneous functions to derive the "virial identity", also known as the Lagrange-Jacobi identity: $d^2(r^2)/dt^2 = 4H + 2U$. Also show that $2H + U = H + K$.*

4.2. **McGehee transformation via Energy Balance.** The key property of the potential energy $-U$, as far as McGehee's transformation is concerned, is that it is homogeneous of degree -1: $U(\lambda q) = \lambda^{-1} U(q)$, or

$$q \mapsto \lambda q \implies U \mapsto \lambda^{-1} U.$$

Our guiding principle in deriving the McGehee blown-up equations is to require that the kinetic energy K must scale the same as potential energy. so as to guarantee that the total energy "balance" under scaling and has a scaling law. We call this the principle of "energy balance". Then it must be that $K \mapsto \lambda^{-1} K$. Since K is quadratic in velocities v this implies that velocities v scale according to

$$v = \dot{q} \mapsto \lambda^{-1/2} v.$$

How must time scale? Since $dq \mapsto \lambda dq$ and $v = dq/dt$, we see that for a power law scaling $dt \mapsto \lambda^a dt$ to yield $v \mapsto \lambda^{-1/2} v$ we must have $a = 3/2$. Summarizing, our space-time scaling law must be

$$q \mapsto \lambda q, dt \mapsto \lambda^{3/2} dt, \tag{9}$$

which induces the desired scalings

$$v \mapsto \lambda^{-1/2} v; (U, K, H) \mapsto (\lambda^{-1} U, \lambda^{-1} K, \lambda^{-1} H)$$

Exercise 2. *Show that $q(t)$ solves (6) if and only also $q_\lambda(t) := \lambda q(\lambda^{-3/2} t)$ solves (6). Explain how the exponent $-3/2$ in this transformation-of-paths formula arises from the $+3/2$ in the time part of the scaling law of eq (9). Show how this yields a derivation of Kepler's 3rd law.*

McGehee's genius was to rewrite Newton's equations, as much as is possible, in scale invariant terms. We cannot completely get rid of scale, but we can encode scale in the single size variable $r = \sqrt{\langle q, q \rangle}$ introduced earlier and through which we remove scale from the remaining variables:

$$q = rs \tag{10}$$

$$v = r^{-1/2} y \tag{11}$$

$$dt = r^{3/2} d\tau \tag{12}$$

These relations define the *McGehee transformation* $(q, v; t) \mapsto (r, s, y; \tau)$. Observe that s lies on the unit sphere $r = 1$ in the configuration space,

$$s \in S = S^{dN-1} = \{r = 1\} \subset \mathbb{E}$$

so that (r, s) are spherical coordinates on $\mathbb{E}$. We sometimes refer to s as the shape of the configuration q.

Exercise 3. *Write $'$ for $\frac{d}{d\tau} = r^{3/2}\frac{d}{dt}$. Show that McGehee's transformation transforms Newton's equations (6) to the equations*

$$\begin{aligned} r' &= r\nu \\ s' &= y - \nu s \\ y' &= \nabla U(s) + \frac{1}{2}\nu y \end{aligned} \tag{13}$$

where $\nu = \langle s, y\rangle$. These equations are the McGehee blown-up equations.

In the last equation $\nabla U(s) \in \mathbb{E}$ is the same gradient as in Newton's equation (6), only restricted to points s of the sphere $\{r = 1\}$. The blown-up equations are analytic and extend analytically to the total collision manifold $r = 0$. For $N > 2$ the equations still have singularities due to partial collisions eg $r_{12} = 0$, at which $\nabla U(s)$ still blows up.

Definition 1. *The "extended collision manifold" is the locus $r = 0$ for the blown-up phase space $[0,\infty) \times S \times \mathbb{R}^{dN}$ of McGehee.*

The first of the three blown-up ODEs asserts that the extended collision manifold is an invariant submanifold. On the extended collision manifold the flow is non-trivial , as a glance at the last two equations shows. Away from the extended collision manifold, the blown-up equations are equivalent to Newton's equations. What have we gained by adding this collision manifold?

4.3. **Equilibria!** The first thing one learns in a class in dynamical systems is to look for equilibria. But Newton's equations have no equilibria. N stars cannot just sit there, still, in space. Adding the extended collision manifold through blow-up introduces equilibria. When $N = 3$ these equilibria correspond to the solutions of Euler and Lagrange described above. In the case of general N the equilibria correspond to "central configurations". See Proposition 1 below.

Finding the equilibria. From the first of the blow-up equations (13) we see that an equilibrium must lie on the extended collision manifold $r = 0$ (consistent with what we just said about "stars cannot just sit there"). Plugging the second equilibrium equation $0 = y - \nu s$ into the third equation of the blown-up equations (13) yields the "shape equation"

$$\nabla U(s) = -\frac{1}{2}\nu^2 s.$$

Taking the inner product of both sides of the shape equation with s and using Euler's identity for homogeneous functions yields

$$U(s) = \frac{1}{2}\nu^2$$

or $\nu = \pm\sqrt{2U(s)}$. Now the gradient of the function r^2 at a point $s \in S$ is $2s$ so that we can rewrite the shape equation as

$$\nabla U(s) = c\nabla(r^2), \qquad c = -\frac{1}{4}\nu^2 = -\frac{1}{2}U(s).$$

Think of c as a Lagrange multiplier. We have proved that the shape s of an equilibrium configuration must be a "central configuration" where:

Definition 2. *A central configuration is a shape $s \in S$ which is a critical point of U restricted to the sphere $r = 1$.*

Conversely, for each central configuration shape $s_{cc} \in S$ we obtain an equilibrium point $(r, s, y) = (0, s_{cc}, y)$ by setting $y = \nu s_{cc}$. with $\nu = \pm\sqrt{2U(s_{cc})}$. We have established

Proposition 1. *Equilibria of the blown-up equation are in 2:1 correspondence with central configurations. This correspondence associates to a given central configuration s_{cc} the two equilibria $(r, s, y) = (0, s_{cc}, \nu s_{cc})$, with $\nu = \pm\sqrt{2U(s_{cc})}$.*

There is another way to arrive at central configurations in keeping with our original discussion of the Euler and Lagrange solutions. Make the ansatz:

$$q(t) = \lambda(t)s \tag{14}$$

where $\lambda(t)$ is a time dependent scalar and $s \in S$ is constant.

Exercise 4. *Show that the ansatz (14) satisfies Newton's equations if and only if s is a central configuration and $\lambda(t)$ satisfies the "Kepler problem":*

$$\ddot{\lambda} = -\mu\lambda/|\lambda|^3 \qquad \mu = \frac{1}{2}\nu^2 = U(s) \tag{15}$$

All solutions of the one-dimensional Kepler problem eq. (15) end in collision $\lambda = 0$ which corresponds to total collision under our anatz (14). The ansatz with real scalar $\lambda(t)$ yields the Lagrange and Euler solutions to the three-body equations which we first described above in terms of "dropping" bodies and having their shape remain constant.

4.3.1. *The Euler and Lagrange family. Planar problems.* Assume we are in the planar case $d = 2$. Identify $\mathbb{R}^2$ with $\mathbb{C}$: $(x, y) \to x + iy$ so that $\mathbb{E} = \mathbb{C}^N$ and so that complex scalar multiplication of $s = (s_1, \ldots, s_N) \in S \subset \mathbb{C}^N$ by $\lambda \in \mathbb{C}$ corresponds to scaling the N-gon s by the factor $|\lambda|$ while rotating it by $Arg(\lambda)$.

Exercise 5. *Show that* $\nabla U(\lambda q) = \frac{\lambda}{|\lambda|^3}\nabla U(q)$

Exercise 6. *Use exercise 5 to show that exercise 4 also holds in the case of the planar N-body problem, with now $\lambda(t) \in \mathbb{C}$ a complex scalar.*

The solutions of exercise 6 are motions in which the N curves in the plane $q_a(t), a = 1, \ldots, N$ are all "homographic" to each other, meaning related by a fixed scaling and rotation. Indeed $\lambda(t)$ describes a conic and $q_a(t) = \lambda(t)s_a$ are all homographic to this single conic, the s_a being the homography factor. We now have, for each *planar* central configuration s, a family of solutions parameterized by the complex solutions $\lambda(t)$ to eq. (15), This family varies from total collision solutions when $\lambda(t) \in \mathbb{R}$ to circular motions when $\lambda(t) = e^{i\omega t} \in S^1 \subset \mathbb{C}$. For fixed energy h we can think of the parameter of the family as being the angular momentum J discussed below. $J = 0$ corresponds to the total collision solution while the maximum and minimum values of J (at fixed H) are circular motions.

4.3.2. *Aside: An open problem.* The potential U is invariant under rotations and translations. Consequently, the central configurations as we defined them are not isolated, but come in families.

Is the set of central configurations, modulo rotatations and translations, a finite set? This problem is attributed to Chazy [7]. See Albouy-Cabral [2] for perspective and a recent survey.

What is known. Some History. $N = 3$: Euler and Lagrange had established the complete list of central configurations as described here. $N = 4$: Simó did some detailed numerical work in [43]. Albouy [1] classified the central configurations in the case of 4 equal masses two centuries two decades and a few years after Euler and Lagrange . One of his main achievements was to show that in the equal mass case the 4-body central configurations all have a reflectional symmetry. Eleven years after Albouy's work Hampton and Moeckel [15] proved that the central configurations are finite (less than 1856) $N = 5$: In 2012 Albouy and Kaloshin [3] proved that for $N = 5$ and away from an algebraic surface in the parameter space $\mathbb{RP}^4_+$ of mass ratios, the number of central configurations is finite. In 1999 Roberts [38] constructed examples for $N = 5$, but with one of the five masses negative, in which the set of central configurations is infinite, underlining the subtlety of the problem.

4.4. **Linear and angular momentum.** Besides energy, the only known constants of motion for the general N-body problem are the components of the linear momentum

$$P = \Sigma m_a v_a$$

and the angular momentum

$$J(q, v) = \Sigma m_a q_a \wedge v_a \tag{16}$$

These momenta are intimately connected to the fact that the group G of rigid motions acts by symmetries of Newton's equations.

Exercise 7. *$v \in \mathbb{E}$ is orthogonal to the G orbit through $q \in \mathbb{E}$ if and only if $P(v) = 0$ and $J(q, v) = 0$*

4.5. **Center of mass frame.** A well-know argument using the Galilean symmetries of boost and translation and found in any introductory physics text allows us to suppose that all our solutions satisfy the linear constraints

$$P = 0 \text{ and } \Sigma m_a q_a = 0$$

In this case we say that we are in "center of mass frame" and we set

$$\mathbb{E}_{cm} = \{q \in \mathbb{E} : \Sigma m_a q_a = 0\} \cong \mathbb{R}^{d(N-1)}.$$

The infinitesimal generators of the translation action are the constant vector fields $(c, c, \dots, c)$, which we abbreviate as $c\vec{1}, c \in \mathbb{R}^d$. We will call the span of these vector fields the translation space. We compute that $\langle c\vec{1}, q\rangle = c \cdot \Sigma m_a q_a$ showing that $\mathbb{E}_{cm}$ is the orthogonal complement to the translation space and consequently realizes the quotient of $\mathbb{E}$ by translations. Observe that the total collision space agrees with the translation space. Thus the only total collision point within $\mathbb{E}_{cm}$ is the origin.

We can go to center of mass frame before or after blow-up, the result is the same, namely the system of ODEs (13) restricted to the subvariety

$$(r, s, y) = [0, \infty) \times S_{cm} \times \mathbb{E}_{cm} \subset [0, \infty) \times S \times \mathbb{E} \tag{17}$$

where

$$S_{cm} = \{q \in \mathbb{E}_{cm} : \langle s, s\rangle = 1\} \cong S^{d(N-1)-1}.$$

4.6. **Energy-momentum level sets and the Standard Collision Manifold.** Because energy and angular momentum are invariant as we flow according to Newton, by fixing their values h and J_0 we obtain invariant submanifolds of phase space:

$$M^{int}(h) = \{H = h, r > 0\}$$

and

$$M^{int}(h, J_0) = \{H = h, J = J_0, r > 0\}$$

Energy and angular momentum are not defined at $r = 0$ so we have excluded $r = 0$. Set

$$M(h) = \text{Closure}(M^{int}(h)), \qquad M(h, J_0) = \text{Closure}(M^{int}(h, J_0)), \tag{18}$$

the closure being within the blown-up phase space. (The superscript "int" is for "interior".) We will need to understand the boundaries of these spaces, which is their intersection with the extended collision manifold $r = 0$; in other words we must understand how these invariant submanifolds approach the extended collision manifold $\{r = 0\}$ as $r \to 0$.

The following notation will be useful in this endeavor.

Definition 3. **[Notation]** *For $F = F(q, v)$ a homogeneous function on $\mathbb{E} \times \mathbb{E}$ write $\tilde{F}$ for the scale-invariant version of F achieved by multiplying F by $r^{-\alpha}$ where α is the degree of homegeneity of F with respect to our weighted scaling. Thus: $F(q, v) = r^{\alpha}\tilde{F}(s, y)$.*

According to "energy balance" both the potential energy, kinetic energy, and total energy are homogeneous of degree -1. Thus

$$\tilde{U}(s) = rU(q)$$

where $\tilde{U}$ is homogeneous of degree 0 and can be viewed as a function on the sphere S_{cm}. And

$$\tilde{H} = rH \tag{19}$$

where

$$\tilde{H}(s, y) = \frac{1}{2}\langle y, y\rangle - U(s) = \tilde{K}(y) - \tilde{U}(s)$$

and $\tilde{K}, \tilde{U}$ are homogeneous of degree 0. The angular momentum is homogeneous of degree 1/2 so that

$$J = r^{1/2}\tilde{J}(s, y) \tag{20}$$

where $\tilde{J}$ is scale invariant and equals $\Sigma m_a s_a \wedge y_a$.

If follows immediately from eq (19) that

$$\partial(M(h)) = \{\tilde{H} = 0, r = 0\}.$$

while using in addition eq (20) we see that

$$\partial(M(h, 0)) = \{\tilde{H} = 0, \tilde{J} = 0, r = 0\}.$$

We give these submanifolds separate names.

Definition 4. *The full collision manifold is* $M_0 = \{\tilde{H} = 0, r = 0\}$.

Definition 5. *The "standard collision manifold" is the locus*

$$C := \{r = \tilde{H} = \tilde{J} = 0\}.$$

Exercise 8. *Show that* M_0 *and* C *are invariant submanifolds of the blown-up flow by using eq (13) to show that*

$$\frac{d}{d\tau}\tilde{H} = \nu\tilde{H}.$$
$$\frac{d}{d\tau}\tilde{J} = -\frac{1}{2}\nu\tilde{J}$$

hold everywhere on the blown-up phase space.

Thus the extended collision manifold contains the full collision manifold M_0 which in turn contains the standard collision manifold C and these are invariant submanifolds. The equilibria all lie on C.

The following theorem is fundamental.

Theorem 2. *(Sundman) If $r \to 0$ along an honest solution, then $J = 0$ for that solution and hence that solution tends to C as $r \to 0$. Moreover, the solution tends to the subset of equilibria within C.*

Here we are using the hopefully obvious

Definition 6. *An "honest solution" to the blown-up equations is a solution such that $r > 0$.*

The honest solutions are just the reparameterizations of solutions to our original Newton's equations according to the blown-up time.

Remark. The standard collision manifold C is the space most authors refer to when they speak of the "collision manifold" for the N-body problem. Chenciner (see also [8]) argues that the standard collision manifold is the dilation quotient of the N-body phase space.

4.7. **Aside: Parabolic infinity.** Set

$$u = 1/r$$

and view $u = 0$ as a neighborhood of infinity.

Exercise 9. *Show that under the change of variables $r \mapsto u = 1/r$, with s, y unchanged the r' equation becomes*

$$u' = -u\nu$$

with ν as before.

Now $u = 0$ becomes an invariant submanifold for the flow. We have there a kind of dual to the theorem of Sundman above. First we need a definition.

Definition 7. *A solution escapes parabolically to infinity as the Newtonian time $t \to \infty$ if its energy $H = 0$ and if in the limit the unrescaled kinetic energy $K(v)$ tends to zero as $t \to \infty$.*

Theorem 3. *(parabolic) Every solution which escapes parabolically to infinity tends to the subset of equilibria in the blown up variables* (s, y).

There is one important difference to keep in mind now. Solutions with nonzero angular momentum can and do escape parabolically to infinity, while no solutions with angular momentum zero limit to the collision manifold.

5. Quotient by Rotations.

Newton's equations and their McGehee blow-ups (eq 13) are invariant under the group G of rigid motions and so descend to ODEs on the quotient space of their phase spaces by G. Working on this quotient instead of the original helps our intuition enormously in the case $N = 3$ and $d = 2$. We describe the quotient and some aspects of the quotient flow for general N and $d = 2$.

The group G of rigid motions is the product of two subgroups, the translation group and the rotation group. We have already formed the quotient of phase space by translations when we went to center-of-mass frame, i.e. by restricting to $s, y \in \mathbb{E}_{cm}$. To form the remaining quotient by rotations it is much cleaner to restrict to the planar case $d = 2$. *Henceforth we assume that we are working with the planar N-body problem,* $d = 2$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ as before. Thus $\mathbb{E} \cong \mathbb{C}^N$ and $\mathbb{E}_{cm} \cong \mathbb{C}^{N-1}$. Represent rotations as unit complex scalars $u \in S^1 \subset \mathbb{C}$ acting on $(q, v) \in \mathbb{E}_{cm} \times \mathbb{E}_{cm}$ by $(q, v) \mapsto (uq, uv)$ and on McGehee coordinates by $(r, s, y) \mapsto (r, us, uy)$.

Definition 8. *The blown up reduced phase space in the planar case is the quotient of the blown-up center of mass phase space* $[0, \infty) \times S_{cm} \times \mathbb{E}_{cm} \cong [0, \infty) \times S^{2N-3} \times \mathbb{C}^{N-1}$ *by the group of rotations. Upon deleting the collision locus* $\mathcal{C}$ *we denote this quotient by*

$$\mathcal{P}_N = ([0, \infty) \times (S^{2N-3} \setminus \mathcal{C}) \times \mathbb{C}^{N-1})/S^1.$$

Let us begin to try to understand this quotient by momentarily forgetting the velocities (v or y) and the fact that we deleted the collision locus $\mathcal{C}$. The circle action sends a blown-up configuration (r, s) to (r, us), $s \in S_{cm}$. So we need to understand the quotient of the sphere $S_{cm} = S^{2N-3}$ by this action of S^1. It is well known that this quotient S_{cm}/S^1 is isomorphic to the complex projective space $\mathbb{CP}^{N-2} := \mathbb{P}(\mathbb{E}_{cm})$ with the projection map $S_{cm} \to S_{cm}/S^1$ being the Hopf fibration. Hence the quotient of the (r, s) by S^1 yields $[0, \infty) \times \mathbb{CP}^{N-2}$.

To better understand the meaning of points of $\mathbb{CP}^{N-2}$, work with a general $q \in \mathbb{E}_{cm}$, not necessarily a unit length vector. We insist only that $q \neq 0$ and allow the scalar u to vary over the larger group $\mathbb{C}^* \supset S^1$ of *all* nonzero complex numbers. The resulting quotient is well-known to be $(\mathbb{C}^{N-1} \setminus \{0\})/C^* = \mathbb{CP}^{N-2}$. The action of $u \in \mathbb{C}^*$ on $q \in \mathbb{C}^{N-1} \setminus \{0\}$ is precisely the action of rotating and *scaling* the (centered) N-gon q.

Definition 9. *The projective space* $\mathbb{CP}^{N-2} = \mathbb{P}(\mathbb{E}_{cm})$ *just constructed is called shape space. Its points represent oriented similarity classes of planar N-gons.*

We have realized the configuration part of the quotient after blow-up as $[0, \infty) \times \mathbb{CP}^{N-2}$ where $\mathbb{CP}^{N-2}$ is the shape space. *When* $N = 3$ *the shape space is the shape sphere described above.*

Collision locus.

The condition that a configuration $q = (q_1, \ldots, q_N)$ represent a collision is that $q_a = q_b$ for some $a \neq b, 1 \leq a, b \leq N$. This condition is complex linear when viewed in homogeneous coordinates $[q_1, q_1, \ldots, q_N]$ and so defines a complex hyperplane, a $\mathbb{CP}^{N-3} \subset \mathbb{CP}^{N-2}$. There are $\binom{N}{2}$ pairs (a, b) and so we have to delete $\binom{N}{2}$ hyperplanes from our shape space. The union of these hyperplanes, viewed projectively, is the collision locus:

$$\mathcal{C} = \{[q] = [q_1, q_2, \ldots, q_N] \in \mathbb{CP}^{N-2} : q_a = q_b \text{ some } a \neq b\}.$$

We use the same symbol for the collision locus before or after quotient.

Accounting for velocities. In the last few paragraphs above we dropped the velocity y. The quotient map $(r, s) \mapsto (r, [s])$ from $[0, \infty) \times S_{cm} \to [0, \infty) \times \mathbb{CP}^{N-2}$ expresses $[0, \infty) \times S_{cm}$ as a principal S^1 bundle over $[0, \infty) \times \mathbb{CP}^{N-2}$.

Now include the velocity y. The quotient procedure with y included is precisely the procedure used to construct an associated vector bundle to a principal bundle. (See for example [18] or [45].) Realizing this, we see that the quotient $\mathcal{P}_N$ is a complex vector bundle over $[0, \infty) \times (\mathbb{CP}^{N-2} \setminus \mathcal{C})$ whose rank is $N - 1$, its fiber being parameterized by $y \in \mathbb{E}_{cm} \cong \mathbb{C}^{N-1}$. What is this vector bundle?

Proposition 2.

$$\mathcal{P}_N = [0, \infty) \times T(\mathbb{CP}^{N-2} \setminus \mathcal{C}) \times \mathbb{R}^2$$

as a vector bundle over $[0, \infty) \times (\mathbb{CP}^{N-2} \setminus \mathcal{C})$*. The final* $\mathbb{R}^2$ *factor can be globally coordinatized by* $(\nu, \tilde{J})$ *where* $\nu = \langle s, y\rangle$ *represents the time rate of change of size and where* $\tilde{J} = \langle is, y\rangle$ *is also equal to* $r^{-1/2}J$ *off of* $r = 0$ *where* J *is the usual total angular momentum of the system. The fiber variable tangent to shape space* $\mathbb{CP}^{N-2}$ *represents "shape" velocity.*

In the case of $N = 3$ we have $\mathbb{CP}^{N-2} = \mathbb{CP}^1 = S^2$, the shape sphere previously discussed in section 3. Then

$$\mathcal{P}_3 = [0, \infty) \times T(S^2 \setminus \mathcal{C}) \times \mathbb{R}^2 = [0, \infty) \times \mathbb{R} \times (S^2 \setminus \mathcal{C}) \times \mathbb{R}^3,$$

where

$$\mathcal{C} = \{B_{12}, B_{23}, B_{31}\}$$

is the set of three binary collision points.

5.0.1. *Velocity (Saari) decomposition.* Passing through a configuration $q \in \mathbb{E}_{cm}$ we have two group-defined curves: the scalings $\lambda q, \lambda \in \mathbb{R}$ of q and the rotations $uq, u \in S^1$ of q. The tangent spaces to these curves are orthogonal, and together with the orthogonal complement of their span they define a geometric splitting of

$T_q\mathbb{E}_{cm} = \mathbb{E}_{cm}$

$$T_q\mathbb{E}_{cm} = (\text{scale}) + (\text{rotation}) + (\text{horizontal}) \tag{21}$$

$$= \mathbb{R}q \oplus i\mathbb{R}q \oplus \{v : J(q,v) = 0, \nu(q,v) = 0\} \tag{22}$$

where

Definition 10. *The horizontal space at q is the orthogonal complement (rel. the mass metric) of the sum of first two subspaces $\mathbb{R}q$ and $i\mathbb{R}q$, i.e it is the orthogonal complement to the $\mathbb{C}$-span of q.*

Refer to exercise 7 and the definition of ν to see why the horizontal space at q is the zero locus of $J(q,v)$ and $\nu(q,v)$.

Unit vectors spanning the scale and rotation spaces are s and is. Consequently, if we take a $v \in T_q\mathbb{E}_{cm}$ and decompose it accordingly we get

$$v = \langle s, v\rangle s + \langle is, v\rangle is + v_{hor} \tag{23}$$

and the scale invariant version:

$$y = \nu s + \tilde{J} is + y_{hor}; \qquad \nu = \langle s, y\rangle, \tilde{J} = \langle is, y\rangle \tag{24}$$

where the subscript "hor" on v and y denote their orthogonal projections onto the horizontal subspace.

Remark. Saari [41] pointed out the importance of the horizontal-vertical splitting of eq (23) in celestial mechanics. This splitting is thus often called the "Saari decomposition" in the context of the N-body problem.

5.0.2. *Proof of proposition 2.* The decomposition (eq (24)) of y is S^1-equivariant. The coefficients of the first two terms ν and $\tilde{J} = \langle is, y\rangle$ are S^1-invariant functions and so are well defined functions on the quotient $\mathcal{P}_N$. The horizontal term y_{hor} , as y varies at fixed s, sweeps out the horizontal subspace at s and these subspaces, as s varies, forms the horizontal distribution associated to a connection on the principal S^1-bundle $S_{cm} \to \mathbb{CP}^{N-2}$. It is a basic fact about principal G-bundles with connection that the union of the horizontal spaces for the connection forms a G-equivariant vector bundle over the total space, and the quotient of this vector bundle by G is canonically isomorphic to the tangent space to the base space. Writing $[s, y]$ to denote the S^1-equivalence class of the pair (s, y) we see that the set of all $[s, y_{hor}]$'s forms $T\mathbb{CP}^{N-2}$. Now s, together with $(y_{hor}, \nu, \tilde{J})$ determine y uniquely. It follows that the map $[s, y] \mapsto ([s, y_{hor}], (\nu, \tilde{J}))$ is a vector bundle isomorphism between the vector bundles $(S_{cm} \times \mathbb{E}_{cm})/S^1$ and $T\mathbb{CP}^{N-2} \times \mathbb{R}^2$ over $\mathbb{CP}^{N-2}$. The radial scaling coordinate r goes along for the ride, without any change. QED

Because the decompositions of equations (23, 24) are orthogonal and the second decomposition is scale invariant it follows that total kinetic energy decomposes as

$$\begin{aligned} K(q, v) &= \frac{1}{2}\frac{\nu^2}{r} + \frac{1}{2}\frac{J^2}{r^2} + \frac{K_{shape}([s, y_{hor}])}{r} \\ &= \frac{1}{r}(\frac{\nu^2}{2} + \frac{\tilde{J}^2}{2} + K_{shape}). \end{aligned} \tag{25}$$

The final term K_{shape} is formed by computing the squared length of the horizontal factor y_{hor} and is canonically identified with the kinetic energy of the standard Fubini-Study metric on the shape space $\mathbb{CP}^{N-2}$.

REMARK. The kinetic energy decomposition (25) shows that for $J \neq 0$ the manifolds $M^{int}(H_0, J)$ is already closed in $\mathcal{P}_N$ so that

$$M(h, J) = M^{int}(h, J) \tag{26}$$

Indeed, the energy equation $rh = \tilde{H}$ shows that $\tilde{U} \geq \frac{1}{2}J^2/r + O(r)$ holds on $M^{int}(h, J)$ which shows that if for a sequence $p_i \in M^{int}(H_0, J)$ we have that $r(p_i) \to 0$ then $U(s_i) \to \infty$ so that the shape s_i of these points p_i are converging to the collision locus $\mathcal{C} \subset \mathbb{CP}^{N-2}$ on the shape space. But we deleted $\mathcal{C}$ in forming $\mathcal{P}_N$.

5.1. **Euler-Lagrange family in reduced coordinates.** What does a planar central configuration family (section 4.3.1) look like in the blown-up reduced coordinates of $\mathcal{P}_N$? We largely follow the exposition of Moeckel [28], section 2.

Let s_{cc} be a planar central configuration and $[s_{cc}] \in \mathbb{CP}^{N-2}$ the corresponding point in shape space. For the associated central configuration family, the shape does not change. In particular the shape velocity $y_{hor} = 0$. Thus among our full set of variables $(r, [s, y_{hor}], (\nu, \tilde{J}))$ of $\mathcal{P}_N = [0, \infty) \times T(\mathbb{CP}^{N-2} \setminus \mathcal{C}) \times \mathbb{R}^2$ we have $[s, y_{hor}] = [s_{cc}, 0]$ being constant, and only the variables $(r, \nu, \tilde{J})$ change along the solution curves of the family. This size r and angle θ of the curves in the family specify the homography factor $\lambda = \lambda(t) = re^{i\theta}$ where $\lambda(t)$ solves the Kepler problem as per exercise 6.

Since the shape does not change, the shape velocity y_{hor} is identically zero along each of these solutions and so $K_{shape} = 0$. Thus along such a solution

$$\tilde{K} = \frac{1}{2}\nu^2 + \frac{1}{2}\tilde{J}^2 = \frac{1}{2}\nu^2 + \frac{1}{2}\frac{J^2}{r}$$

(see eq. (25)) But $\tilde{J} = r^{1/2}J$ and J is constant along solutions so the change of r and choice of J determines the change of $\tilde{J}$. So we can think of the only variables for the family as being ν, r.

Fix the energy h. We can then view the central configuration family as a one-parameter family of curves in the (ν, r) plane, the parameter being the angular momentum J. Indeed the energy equation reads:

$$rh = \frac{1}{2}\nu^2 + \frac{1}{2}J^2/r - U(s_{cc}).$$

and since $U(s_{cc})$ is constant, this defines a one-parameter family of curves. We plot these curves in the ν, r plane for various values of the angular momentum J below in figure 5.1.

Observe the rest point cycle in this picture: the closed curve passing through the two equilibria. This curve is the union of two solution curves, a top arch which is an honest solution, and a bottom return curve lying in M_0. The top arch lies on

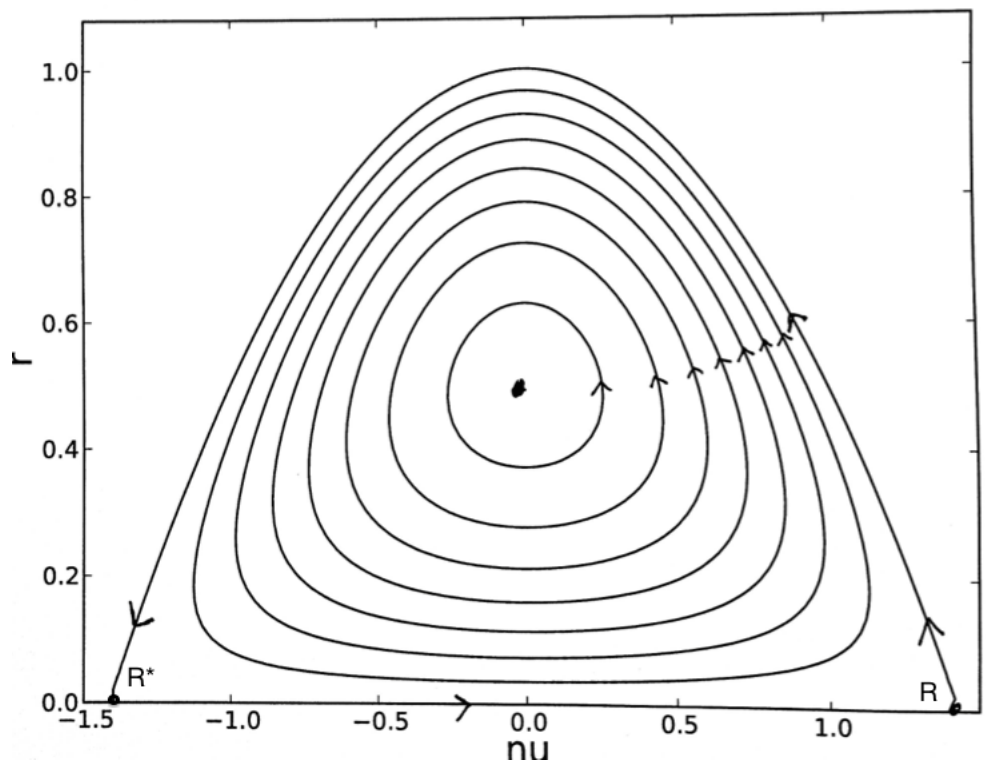


FIGURE 7. A central configuration family in ν, r coordinates. The arch and 'floor' $r = 0$ comprise the rest cycle

$M(h, 0)$ [4]. and is the ejection-collision orbit described when we introduced central configurations in section 2.2 by describing Lagrange's solution: the orbit explodes out of total collision along the shape s_{cc} achieves a maximum size and shrinks back to triple collision. It connects the rest point $R \in C$ having shape s_{cc} and $\nu = \sqrt{2U(s_{cc})} > 0$ with the rest point R_* having shape s_{cc} and $\nu = -\sqrt{2U(s_{cc})} < 0$. The bottom 'return road' lies on the full collision manifold $\{r = 0, \tilde{H} = 0\} = M_0$ and yields a return route from R_* to R.

This rest point cycle is the limit of the family of the periodic central configuration solutions with $J \neq 0$ as $J \to 0$. It is important for later on that the return road

[4] **Notational Convenience.** We have just used the symbol $M(h, 0) \subset \mathcal{P}_N$ for what used to be a submanifold of the phase space *before quotient.* We will continue to use the same notation for any G-invariant submanifold or function on phase space before or after the quotient procedure. Thus we have:

$$C, M_0, M(h), M^{int}(h), M(h, J_0), \text{ etc. } \subset \mathcal{P}_N.$$

DOES NOT lie on the standard collision manifold C, and that it does not consist of rest points.

6. A gradient-like flow!

A flow is called "gradient-like" if it admits a continuous function f, which we call a 'Liapanov function' which is strictly monotone decreasing along all solution curves except for the equilibria. (See Robinson [39] p. 357.) The dominant aspect of the flow on the full collision manifold M_0 is that it is gradient-like with $-\nu$ as Liapanov function. See Proposition 3 below.

Exercise 10. *Use equations (13) to derive the identity*

$$\nu' = \tilde{K} - \frac{1}{2}\nu^2 + \tilde{H} \tag{27}$$

(See for example Moeckel [24]*, eq. (1.6).)*

Exercise 11. *Use the "Saari decomposition" of kinetic energy (eq (25)) to show that*

$$\tilde{K} - \frac{1}{2}\nu^2 = K_{shape} + \frac{1}{2}\tilde{J}^2.$$

Conclude, using the previous exercise, that

$$\nu' = K_{shape} + \frac{1}{2}\tilde{J}^2 \geq 0 \text{ on } M_0 = \{r = 0, \tilde{H} = 0\}. \tag{28}$$

You have proved much of

Proposition 3. $\nu' \geq 0$ *everywhere on the full collision manifold* M_0*. Moreover* ν *is constant along a solution lying in* M_0 *if and only if that solution is one of the equilibria.*

Remark.

Proof of proposition 3. In exercise 11, eq (28) you proved that $\nu' \geq 0$. It remains to show that any solution which lies on the locus where $\nu' = 0$ is in fact an equilibrium. Eq 28 implies that $\nu' = 0$ if and only if $K_{shape} = 0$ and $\tilde{J} = 0$.

Now $K_{shape}([s, y]) = 0$ if and only if $y_{hor} = 0$. So, $\nu' = 0$ if and only if both y_{hor} and $\tilde{J} = 0$. But then the only nonzero term in the Saari decomposition of eq 24 is the real term so that $y = \lambda s$ with $\lambda \in \mathbb{R}$. Take inner products with s to find that $\lambda = \nu$, or $y - \nu s = 0$ for any such point. Assume now that we have a solution curve $(s(\tau), y(\tau))$ of such points in M_0 lying on the locus $\nu' = 0$. Differentiating the equation $y(\tau) = \nu(\tau)s(\tau)$ using the blow-up equations we see that $y' = \nu' s + \nu s'$. But $\nu' = 0$ by assumption and $s' = y - \nu s = 0$ by the blow-up equations, so $y' = 0$ along the solution: our curve is an equilibrium.

QED

Example 1. Return the central configuration family $s = s_{cc}$ described in the earlier subsection 5.1 and its limiting "return path" on M_0 indicated in figure 5.1. From $rh = \tilde{K} - U$, $U = U(s_{cc})$, and $K_{shape} = 0$ we have that $0 = \frac{1}{2}\nu^2 + \frac{1}{2}\tilde{J}^2 - U(s_{cc})$ or $\frac{1}{2}\nu^2 + \frac{1}{2}\tilde{J}^2 = U(s_{cc})$ for this return path. On the other hand $\nu' = K_{shape} + \frac{1}{2}\tilde{J}^2 = \frac{1}{2}\tilde{J}^2$. These equations yield two important conclusions : (1) the only rest points on the family are at C where $\tilde{J} = 0$. and (2) as we approach points along the return path from the interior $M^{int}(h)$ we have that $J \to 0$ and $r \to 0$ in such a way that $\tilde{J} = r^{-1/2} J$ tends to a finite nonzero limit.

6.1. Making Moeckel's manifold with corner into a manifold with a T.

In [28], at the beginning of section 2, Moeckel constructs a certain manifold with corners in preparation for perturbing the heteroclinic tangles lying on $M(h, 0)$ into the realms of $M(h, \epsilon)$. (He denotes his manifold with a corner by M_{0+} and later simply M.) Dynamics on this manifold-with-corners is essential to our proof of theorem 1. I had a hard time making sense of this manifold so I rederived what Moeckel did in a slightly different way. I get a "manifold with a T" instead of Moeckels manifold with a corner. A "T " is made out of two corners. One corner is Moeckel's manifold with a corner and the other is a reflection of it. The corner itsel is our good friend C, the standard collision manifold. (Figure 6.1 .)

Recall that $M(h)$ is a hypersurface in $\mathcal{P}_N$, and as such is a manifold with boundary, whose boundary is our friend the full collision manifold $M_0 = \{r = 0, \tilde{H} = 0\}$.

Definition 11. $\hat{M}(h) = M(h, 0) \cup M_0 \subset M(h)$.

$\hat{M}(h)$ is a codimension 1 subvariety of the smooth manifold with boundary $M(h)$. It is the zero locus of the function $r\tilde{J}$ restricted to $M(h)$ and as such has two algebraic components : $r = 0$ which is our full collision manifold M_0, and $J = 0$ which forms $M(h, 0)$. The singular locus of $\hat{M}(h)$ is the intersection $C = \{r = 0, \tilde{J} = 0\}$ of these two components. All the rest point cycles described above associated to the central configurations lie on this $\hat{M}(h)$. $\hat{M}(h)$ is comprised of two "manifolds with corners", namely $\{r\tilde{J} = 0, \tilde{J} \geq 0\}$ and $\{r\tilde{J} = 0, \tilde{J} \leq 0\}$. The first of these is Moeckel's manifold with a corner.

$\hat{M}(h)$ is to be viewed as the limit as $J \to 0$ of the manifolds $M(h, J)$.

Proposition 4. *For $S \subset \mathbb{R}$ a subset of the line of angular momentum values, set $M^{int}(h, S) = \cup_{J \in S} M^{int}(h, J)$. Then $\hat{M}(h) = \cap_{\epsilon > 0} M^{int}(h, (-\epsilon, \epsilon))$.*

The proof of the proposition follows in a routine way from our expressions for scaled energy and angular momentum $rh = \tilde{H}$, $J = r^{-1/2}\tilde{J}$ and the kinetic energy decomposition of eq (25). It is useful to recall, eq (26) that the $M(h, J) = M^{int}(h, J)$ are closed for $J \neq 0$.

As an alternative to the description of the proposition, we can either let $J \to 0$ from above or below. Set

$$\hat{M}_+(h) = \lim_{J \to 0^+} M(h, J)$$

and

$$\hat{M}_-(h) = \lim_{J \to 0^-} M(h, J).$$

Then one can show without difficulty that

$$\hat{M}(h) = M_+(h) \cup M_-(h),$$

with $\hat{M}_+(h) = \{p \in \hat{M}(h), \tilde{J} \geq 0\}$ and $\hat{M}_-(h) = \{p \in \hat{M}(h), \tilde{J} \leq 0\}$ being the two manifolds with corners described earlier, Moeckel's manifold with a corner being $\hat{M}_+$.

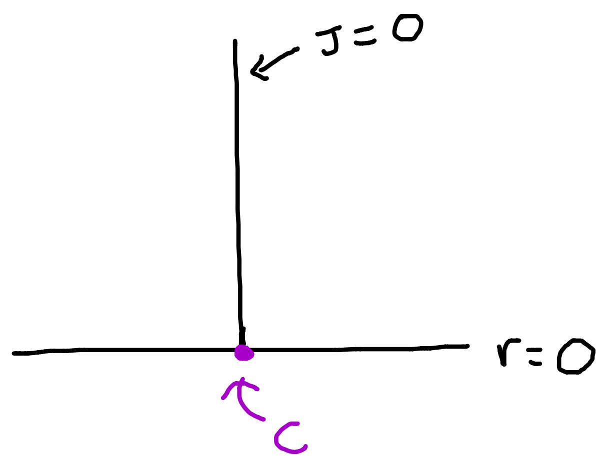


FIGURE 8. $\hat{M}(h)$ inside $M(h)$ is the zero level set of $r\tilde{J}$.

What is a manifold with a 'T'? Suppose we have two real-valued functions x, y on an n-dimensional manifold Q such that 0 is a regular value for both functions and $(0, 0)$ is a regular value of the map $(x, y) : Q \to \mathbb{R}^2$. Then the locus $\{xy = 0, y \geq 0\}$ is a manifold with a T. Its singular locus is $\{x = y = 0\}$. A manifold with a T is locally diffeomorphic to the product of the "upside down T" $xy = 0, y \geq 0$ in the xy plane, by an $\mathbb{R}^{n-2}$. See figure 8.

6.2. **Finishing up the proof of theorem 1.** The idea of Moeckel is that hyperbolic structures persist on perturbation, and that the various stable-unstable connections between Euler and Lagrange central configuration points on $\hat{M}(h)$ are sufficiently "hyperbolic" that they persist into $M(h, \epsilon)$ for $\epsilon \neq 0$ small. Nonzero angular momentum is needed to get orbits connecting from R's to R^* in finite time since the rest cycle of figure 5.1 takes infinite blown-up time. Moeckel cannot carry out the 'perturbation of hyperbolic" idea literally because he cannot establish the needed hyperbolicity or transversality. Instead, following an earlier idea of Easton, he replaces hyperbolicity by a weaker notion of "topologically transverse" between collections of "windows" transverse to the flow. This notion is sufficiently flexible and stable to allow Moeckel to perturb the various formal connections to get actual orbits realizing walks in the abstract graph introduced in section 3. By following the details of his proof, three decades later, we were able to verify that his realizing solutions when projected onto the shape space do indeed stay C^0-close to the concrete connection graph as described in section 3.

The hypothesis of equal or near equal masses is needed to insure that (some of) the eigenvalues for the linearization at the Euler equilibria are complex. This complexity implies a "spiralling" of the Lagrange stable/unstable manifolds around the Euler unstable/stable manifolds and is needed to insure that all connections in the abstract connection graph are realized.

7. A conjecture. Non-existence.

Theorem 1 asserts the existence of a family of small-angular momentum solutions which realize any given free homotopy class. What about our original problem, described in subsection 1.1, of realizing clases for the angular momentum zero three-body problem? The simplest classes of all are those which wind once around a binary collision. They are represented by a curve in which two of the masses rotate once around their common center of mass while the third body remains motionless, far away. We call these classes "tight binary classes". Their syzygy sequence is ij where i, j are the two moving masses.

Conjecture 1. *There is no reduced-periodic solution to the equal-mass zero angular momentum three-body problem which realizes a tight binary class.*

We present four pieces of evidence supporting the conjecture.

1. Hyperbolic Pants.
2. (Not) Hanging out at Infinity.
3. Danya Rose's Bestiary.
4. Failure of limits.

7.1. **Hyperbolic Pants.** Continue to take the masses all equal, but change the potential from Newton's $1/r$ potential to the 'strong-force' $1/r^2$ - the same one investigated by Poincaré [**?**]. I proved in [**?**] that the tight binary class is not realized by a reduced periodic solution of this modified problem.

7.2. **Hanging out at Infinity.** I tried to establish existence of tight-binary type zero angular momentum periodic solutions of "Earth-Moon-sun type" using perturbation theory, following Meyer [21] as a guidebook. If they existed these would be solutions in which 1 and 2 are far away, moving counterclockwise in an approximate circle about their common center of mass while that center of mass moves clockwise in a slow circle about mass 3. Such a motion could not realize a single tight binary as a periodic solution in the inertial frame since there are many months (1-2 circles) in a single slow year. But we only care about relative motion, so it looks conceivable that such a motion could be reduced periodic, executing $1-2$ in a single reduced period.

Connor Jackman, a UCSC graduate student, recently proven that this approach cannot succeed [17]. To explain his result, we begin by investigating the relevant Hill region. Fix the angular momentum to be zero and the energy to be some negative constant. Project this codimension two energy-momentum level set onto shape space to achieve the Hill region. If U is the negative of the potential so that $H = K - U$ with K the kinetic energy and if we set $H = -h < 0$ then the Hill region is the region for which $U \geq h$. The result is depicted in figure 9. The radial coordinate R in that picture is a measure of the overall size of the triangle. From the picture we see that imposing a constraint of the form $R > R_0$ for R_0 large enough, breaks the Hill region into three components. Each component is associated to a partition of the three bodies into a tight binary configuration and a far mass. Insisting that $R >> 1$ is equivalent to insisting that in each component one of the two distances is much smaller than the other two. That is, insisting $R >> 1$ is the same as saying that we are working within the realm of perturbation theory.

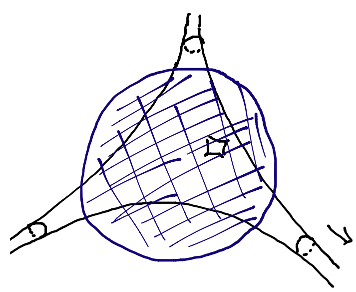


FIGURE 9. Cutting a ball from a pair of pants.

We can now state Jackman's theorem

Theorem 4. *Fix angular momentum zero, total energy to some negative constant. Then there is an* $R_0 > 0$ *(large) such that any solution (periodic or not!) which begins in the region* $R > R_0$ *must enter the region* $R < R_0$.

Jackman's theorem holds for any angular momentum, for any mass distribution as long as the masses are all positive and any negative energy. The F_0 will depend on the masses, the angular momentum, and the energy.

As a corollary to the theorem, we see that there is no orbit having the pattern described at the beginning of this section.

7.3. **The Bestiary of Danya Rose.** Danya Rose is a recent PhD [2015] from Syndey, Australia, who worked under the direction of Holger Dullin. The heart of his thesis is a systematic, very detailed numerical study of the equal-mass zero angular momentum three-body problem which contains over 300 non-collinear, non-isosceles solutions. These solutions are meticulously laid out in over 700 pages of Appendix F of his thesis. I reproduce two sample pages here, below. He titles this appendix "A bestiary of periodic orbits" . The Bestiary contains no solutions which even come close to representing a simple tight binary.

The pages come in pairs, one of which contains an array of statistics of the solution, and the other consists of four pictures, one being that solution drawn as three curves in inertial space, another begin the corresponding curve on the shape sphere, and a third being the curve viewed on the regularized shape sphere.

I describe some details of his search strategy so you can decide for yourself how convincing the data is. At the end of this subsection, I reproduce two pages from the Bestiary. The search proceeds in two basic steps. The first I call "gravitational billiards" and relies crucially on the fundamental domain defined by the group generated by the discrete symmetries available when the masses are equal. The second step is a careful numerical integration on the regularized shape space.

7.3.1. *Coding Gravitational billiards.* When the masses are equal the problem admits mass interchange $q_i \leftrightarrow q_j$ as a discrete symmetry. These symmetries are involutions and generate the symmetric group on 3 letters acting on configuration space by permuting the position coordinates. On the shape sphere the mass interchange involution $q_i \leftrightarrow q_j$ acts as a half-twist about the binary collision ray $r_{ij} = 0$.

We add to these involutions the reflection about any line in the plane which generates reflection about the equator on the shape sphere. Together these involutions generate a 12 element group, the dihedral group D_6 associated to a regular hexagon, which acts on configuration space, mapping solutions to solutions. We used this group to great advantage in [11].

The D_6 action on configuration space induces an action on the shape space and the shape sphere. A fundamental domain for the action on the shape sphere consists of a spherical triangle whose vertices are a binary collision, say B_{12}, a neighboring Euler point, say E_1, and the 'upper' Lagrange point, L_+. See figure 10 where, following Rose, we relabel these points B, M and E. The three edges are labelled A, O and C. The edges A and O correspond to Acute and Obtuse Isosceles triangles. C represents for collinear triangles.

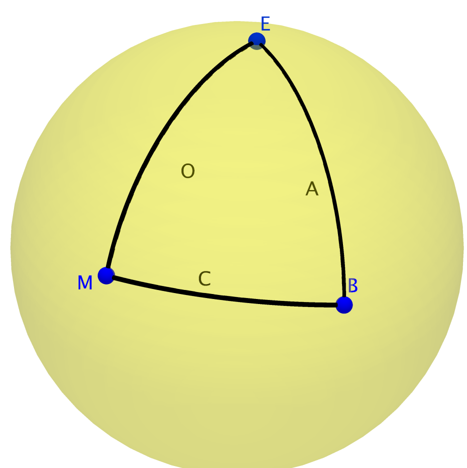


FIGURE 10. A fundamental domain on the shape sphere, with labels.

The corresponding fundamental region on the shape space $\mathbb{R}^3$ is the inverse image of this spherical triangle under central projection, intersected with the Hill region. This shape space fundamental domain is an curvilinear (ideal) tetrahedron. See figure 11. The outer face of this tetrahedron lies on the Hill boundary, labelled

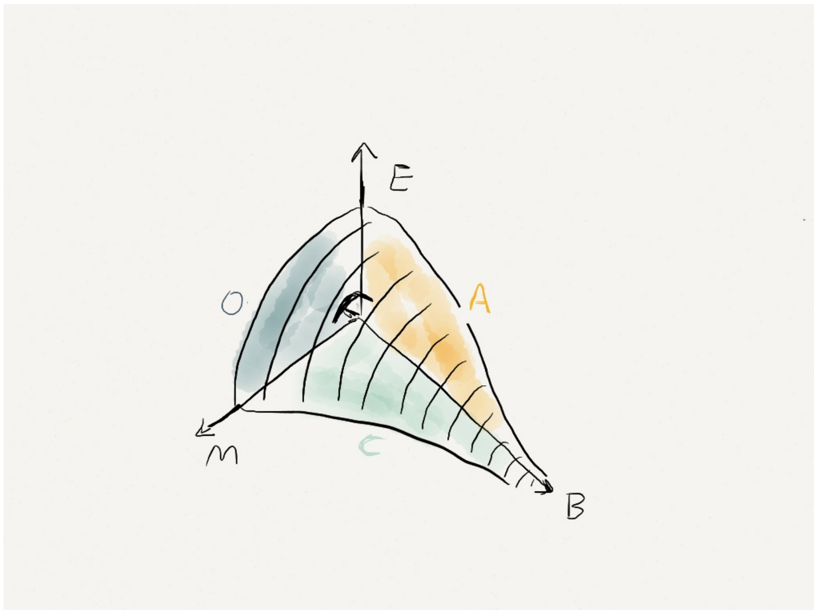


FIGURE 11. A fundamental domain on the shape space is a curvilinear tetrahedron.

"F" by Rose, F being for Freefall. It corresponds to the single face of the spherical triangle under central projection. Three of the four vertices of the tetrahedron lie on this face, one vertex being ideal, at infinity along the binary ray. The other two vertices on this Hill boundary face correspond to the isosceles and Lagrange brake initial conditions. The tetrahedron's fourth vertex is triple collision, at the origin of $\mathbb{R}^3$. The other three faces of the tetrahedron correspond to the three edges of the spherical triangle. The three edges incident to triple collision continue to carry the labels of their corresponding vertices in the spherical triangle.

We summarize Rose's coding of the tetrahedral elements in the following two tables.

TABLE 1. Faces of the fundamental tetrahedron = Edges plus face of fundamental spherical triangle

A	Acute (Isosceles)
O	Obtuse (Isosceles)
C	Collinear
F	Freefall (Hill Boundary)

TABLE 2. Relevant edges of Fundamental Tetrahedron = Vertices of Fundamental spherical triangle

B	Binary
E	Equilateral (= Lagrange)
M	Midpoint (= Midpoint)

A theorem of mine ([33]) asserts that any negative energy zero angular momentum solution must repeatedly intersect "C"- collinearity, and hence cannot stay inside the interior of a Fundamental domain for all time. Since such a solution must hit 'C' we may as well start on C.

Inspired by this theorem, Rose starts off with initial condition on the collinear face, and 'shoots' in to the domain by choosing velocities pointing in. He numerically integrates. That solution enters into the three-dimensional interior table and leaves it through some other face. Instead of leaving, he can reflect that solution back in, using the associated reflection of that face, and continue. Equivalently, each time we hit a face, we apply Snell's law to the associated vector, reflect it back in to the region and continue by applying Newton's equations to this new initial condition. In this way, we get a billiard problem, on the tetrahedral "table". The trajectories bounce off the walls by the standard billiard reflection principle, and move inside the interior according to the zero-angular momentum reduced Newton's equations.

For any such solution, we simply list the tetrahedral boundary elements hit, in the order of hitting, as per the standard practice of symbolic dynamics. This list is a word in the seven letters A,O,C, F, B, M, E. Rose calls this the orbit's "sequence type" The simplest realization of a tight binary would have sequence type COCO.

Remark on Vertices. Following Rose, we ignore the vertices of the tetrahedron in our listing. Here are some good reasons for ignoring the vertices. Two of the vertices lying on the Hill boundary, correspond to the Lagrange and Euler homothety solutions. They represent single known solutions The remaining vertex on F is the ideal one of a binary collision in "free fall" . The corresponding "solution" is a "hard binary": an ideal motion in which two of the masses are eternally stuck together in collision, falling in to the third mass. The final vertex is triple collision, perhaps the most interesting, about which volumes have been written, including the bulk of the present paper. But a periodic solution cannot hit triple collision. Rose cuts off his integrations when they get too close to triple collision, so this vertex is not relevant for Rose's investigations.

We now copy two pages from the Bestiary. Following that we will explain a bit about the regularized shape sphere and the integration method.

From DANYA ROSE's BESTIARY, p. 163 , 164.

7.3.2. *B-mode, Unstable: t0 (8, 5).* Isotropy subgroup: $\{(I, 0)$, $(\tau\rho\sigma_2, \frac{1}{2})$, $(\tau\rho s_1, \frac{1}{4})$, $(\sigma_2 s_1, \frac{1}{4})$, $(s_2, \frac{1}{2})$, $(\tau\rho\sigma_2 s_2, 0)$, $(\tau\rho s_3, \frac{3}{4})$, $(\sigma_2 s_3, \frac{3}{4})\}$

Sequence type: $(A\Omega C\Omega')^4$

Ω: $OCACOACAOACACOA$

$$T_p = 67.01921804$$

$$T_r = 14.96879087$$

$$\Delta G = 12.36682876$$

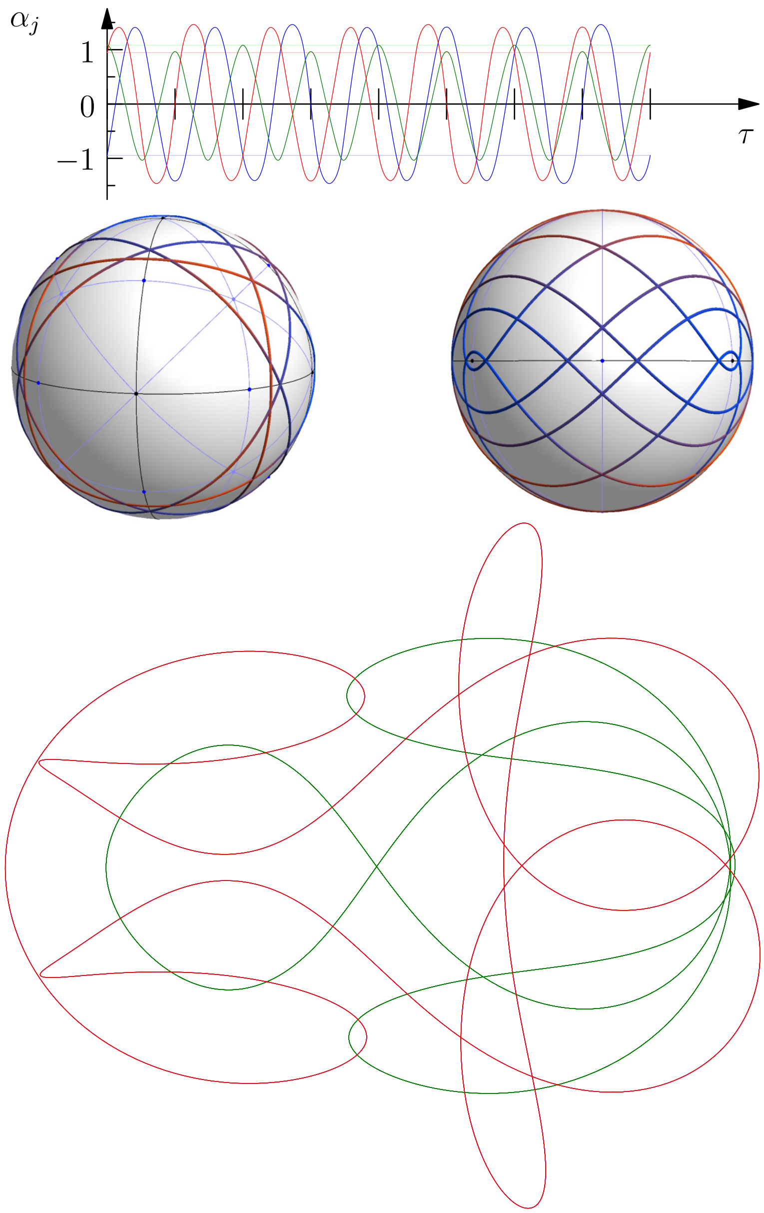
$$\Delta\phi = -8.57712227$$

$$W = 0.00000000$$

$$z_1 = \begin{pmatrix} -0.94542673 \\ +1.08224817 \\ +0.94542673 \\ +1.17170924 \\ -0.00000000 \\ +1.17170924 \end{pmatrix} \qquad z_2 = \begin{pmatrix} -1.41582468 \\ +0.96447581 \\ -0.00000000 \\ +0.00000000 \\ -0.00000000 \\ +1.72244433 \end{pmatrix}$$

$$R_1 = \tau\rho\sigma_2 s_2 \qquad R_2 = \tau\rho s_3$$

$$\lambda = \begin{pmatrix} +25.42672460 + 0.00000000i \\ +1.00000000 + 0.00000000i \\ +0.25229380 + 0.96765068i \\ +0.25229380 - 0.96765068i \\ +1.00000000 + 0.00000000i \\ +0.03932870 + 0.00000000i \end{pmatrix} \qquad |\lambda| = \begin{pmatrix} +25.42672460 \\ +1.00000000 \\ +1.00000000 \\ +1.00000000 \\ +1.00000000 \\ +0.03932870 \end{pmatrix}$$


α_j
1
0
-1
τ

7.3.3. *Regularized Shape Sphere and Numerics.* Binary collisions are a singularity of the ODEs defining the three-body problem and create havoc with numerics. Levi-Civita regularized a single isolated binary collision in the planar problem by using a kind of branched cover over the collision point followed by a time change. Lemaître extended what Levi-Civita did so as to apply to all three binary collisions in a democratic manner. His work was followed up by Murnaghan, Waldvogel who wrote several sets of explicit polynomial regularized equations. See also [31] for a geometric perspective on the regularization maps with many pictures. Rose integrates the reduced regularized equations as written down by Waldvogel, using a symplectic integrator developed along with Dullin. Rose keeps track of when and how solutions cross the various faces using an elegant idea due to Henon for accurately computing Poincaré sections. An added bonus is that in a natural system of coordinates, denoted $\alpha_1, \alpha_2, \alpha_3$, the faces of the regularized fundamental tetrahedron have a very simple description.

7.4. **Failure of limits.** What happens if we take the solutions guaranteed by our theorem 1 and let the angular momentum $J \to 0$? All of them limit to various concatenations of Euler collision-ejection central configuration solutions joined at triple collision. In other words: they all die in total collision.

Moeckel and I spend a few weeks trying to establish other hyperbolic-based 'return mechanisms" in the spirit of figure 7 above which would work for $J = 0$. Such a mechanism would have allowed us to construct the requisite symbolic dynamics and thereby get existence of near-but-not collision periodic solutions having $J = 0$ and the desired reduced syzygy sequences. Our efforts repeatedly failed.

8. Acknowledgements

I am grateful for critique and feedback in the preparing this article from Alain Albouy, Alain Chenciner, Carles Simó, Rick Moeckel, Connor Jackman and Gabriel Martins. Wu-yi Hsiang asked the central question which inspired this research. As described, a conversation with Carles Simo completely redirected my methods to the ones that were ultimately successful. The main result owes its existence to Rick Moeckel. I am thankful to the participants of the CIMAT school in Guanajuato, Mexico, to the organizer of that school Rafael Herrera, to Gil Bor for good Israeli salads, company, and piano playing and to Patricia Carral and Eyal Bor for their hospitality and use of a comfortable bed during this school. After much of this writing was completed I learned from Holger Dullin about Danya Rose's work. The heart of this exposition, the sections on McGehee blow-up were first TeXed up for a seminar I ran with Rafe Mazzeo at Stanford over a decade ago and I would like to express gratitude to Rafe and all the seminar attendees. I would like to thank both Dullin and Rose for a number of email correspondences. Finally I wish to thank NSF grant DMS-1305844 for essential support.

References

[1] A. Albouy, *Symétrie des configurations centrales de quatre corps* C. R. Acad. Sci. Paris, 320, no. 2, (1995), 217-220.

[2] A. Albouy, H. Cabral, and A. Santos, *Some problems on the classical n-body problem*, Celestial Mechanics and Dynamical Astronomy, 113 (2012), 369-375. (available at: http://arxiv.org/abs/1305.3191)

[3] A. Albouy and V. Kaloshin, *Finiteness of central configurations of five bodies in the plane*, Ann. of Math. (2) , 176, no. 1, (2012), 535-588.

[4] G. Birkhoff, **Dynamical Systems**, American Math Society, Colloq Pub, vol. 9, 1927, revised ed. , 1966.

[5] Gil Bor and R. Montgomery, *Poincare y el problem de N-cuerpos*, Miscelanea matematica de la Sociedad Matematica Mexicana, May issue, no. 57 'extraordinario' (2013).

[6] A. Chenciner, *À l'infini en temps fini*, Séminaire Bourbaki vol 39, (1996-1997), 323-353

[7] Chazy, J, *Sur certaines trajectoires du problme des n corps*, Bull. Astron. 35, (1918), 321-389.

[8] A. Chenciner *Collisions totales, mouvements complètement paraboliques et réduction des homothéties dans le Problème des N corps*, Reg. and Chaotic Dyn. , 3 (1998), . 93-106.

[9] A. Chenciner, *Action minimizing solutions of the Newtonian n-body problem: from homology to symmetry,* ICM 2002, v. 3., (2002).

[10] A. Chenciner, preprint website, http://www.imcce.fr/Equipes/ASD/person/chenciner/preprint.php

[11] A. Chenciner and R. Montgomery, *A remarkable periodic solution of the three-body problem in the case of equal masses,* Ann. of Math., v. 152, (2000), 881-901,

[12] A. Chenciner, J. Gerver, R. Montgomery, and C. Sim, , *Simple choreographies of N bodies: a preliminary study*, in **Geometry, Mechanics, and Dynamics**, volume in honor of the 60th birthday of J.E. Marsden, P. Newton, P. Holmes, A. Weinstein, ed. , Springer-Verlag, 2002.

[13] R. Devaney, *Triple collision in the planar isosceles three-body problem*, Inv. Math. **60** (1980), 249–267.

[14] L. Euler, Considérations sur le problème de trois corps, *Mémoires de l'Académie de Sciences de Berlin* **19** , (1770) , 194-220.

[15] M. Hampton and R. Moeckel, *Finiteness of relative equilibria of the four-body problem*, Invent. Math. , 163 , no. 2, (2006), 289- 312.

[16] T. Iwai, A gauge theory for the quantum planar three-body problem. *J. Math. Phys.* **26**, (1987) , 964-974.

[17] C. Jackman, *The Binary Returns!* arxiv:1512.01852, (2015).

[18] S. Kobayashi and K. Nomizu, **Foundations of Differential Geometry**, vol. 1. Interscience pub., 1969.

[19] J-L Lagrange, *Essai sur le Problme des Trois Corps*, Prix de l'Academie Royale des Sciences de Paris, tome IX, 1772; (in vol. 6 of Oeuvres (page 292).)

[20] R. McGehee, *Triple collision in the collinear three-body problem*, Inv. Math, **27** , (1974) 191–227.

[21] K. Meyer, **Periodic solutions of the N-body problem,** Lecture Notes in Mathematics, 1719. Springer-Verlag, Berlin, 1999

[22] J. Moser, **Stable and Random Motion in Dynamical Systems**, Princeton U. Press, 1973.

[23] R. Moeckel, *Orbits of the three-body problem which pass infinitely close to triple collision*, Amer. Jour. Math, v. 103, no. 6, (1981), 1323-1341.

[24] R. Moeckel, *Orbits Near Triple Collision in the Three-Body Problem*, Indiana Univ. Math J, v. 32, no 2, (1983). 221-239.

[25] R. Moeckel, *Heteroclinic phenomena in the isosceles three-body problem*, SIAM Jour. Math. Anal., v. 15, (1984), 857-876.

[26] R. Moeckel, *Some qualitative features of the three-body problem*, in **'Proceedings of the AMS Summer Research Conference on Hamiltonian Systems**, K. Meyer and D. Saari,ed., Contemporary Mathematics 81, AMS (1987), 1-21.

[27] R. Moeckel, *Chaotic Orbits in the Three-body Problem*, in **Periodic Solutions of Hamiltonian Systems and Related Topics**, P.H. Rabinowitz ed., D. Reidel pub., 1987.

[28] R. Moeckel, *Chaotic dynamics near triple collision*, Arch. Rat. Mech, v. 107, no. 1, (1989), 37-69.

[29] R. Moeckel, *Symbolic Dynamics in the planar three-body problem*, Reg. and Chaotic Dyn., v. 12, issue 5, (2007), 449-475.

[30] R. Moeckel and R. Montgomery, *Realizing All Reduced Syzygy Sequences in the Planar Three-Body Problem*, Nonlinearity 28 (2015) 1919-1935 (also: arXiv:1412.2263)

[31] R. Moeckel and R. Montgomery, *Symmetric Regularization, Reduction, and Blow-Up of the Planar Three-Body Problem*, Pac. J. Math. , Vol. 262, No. 1, (2013), 129-189.

[32] R. Montgomery, *The N-body problem, the braid group, and action-minimizing periodic orbits*, Nonlinearity, vol. 11, no. 2, (1998). 363-376.

[33] R. Montgomery, *Infinitely Many Syzygies Archives for Rational Mechanics*, v. 164, no. 4, (2002), 311-340.

[34] R. Montgomery, *The Three-Body Problem and the Shape Sphere*, Amer. Math. Monthly, v 122, no. 4, (2015), 299-321 , (also: arXiv:1402.0841)

[35] C. Moore, *Braids in Classical Gravity*, Physical Review Letters **70**, (1993), 3675–3679

[36] J. Moser, **Stable and Random Motions in Dynamical Systems**, Princeton Univ Press, 1973.

[37] H. Poincaré, *Sur les solutions périodiques et le principe de moindre action*, C.R.A.S. Paris 123 (1896), 915- 918.

[38] G. E. Roberts, *A continuum of relative equilibria in the five-body problem*, Phys. D 127 (1999), 141-145.

[39] C. Robinsion, **Dynamical Systems**, CRC Press, 2nd ed, 1999.

[40] D. Rose, *Geometric phase and periodic orbits of the equal-mass, planar three-body problem with vanishing angular momentum,* PhD thesis, School of Mathematics and Statistics University of Sydney, 2015.

[41] D. Saari, *From rotation and inclination to zero configurational velocity surfaces, I, a natural rotating coordinate system,* Celestial Mechanics, **33**, (1984), 299-318,

[42] C. Simó, *Analysis of triple collision in the isosceles problem*, in **Classical Mechanics and Dynamical Systems**, Marcel Dekker, New York, 1980.

[43] C. Simó, *Relative equilibrium solutions in the four-body problem*, *Celestial Mechanics*, 18, (1978), 165–184.

[44] C. Simó, A. Susín. , *Connections between invariant manifolds in the collision manifold of the planar three-body problem* in T. Ratiu, editor, *The Geometry of Hamiltonian Systems*, pag. 497-518, *MSRI* series, Springer, 1991.

[45] N. Steenrod, **The topology of fiber bundles**, Princeton U. Press, 1951.

[46] A. Wintner, **The Analytic foundations of Celestial Mechanics**, Princeton Univ. Press, 1947.